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Differential Geometry/Group Theory

PseudoRiemannian geometry and actions of simple Lie groups

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Abstract

Let G be a connected noncompact simple Lie group acting isometrically on a connected compact pseudoRiemannian manifold M. Denote with n_0 and m_0 the dimension of the maximal null subspaces tangent to G and M, respectively. Then we always have $n_0 \leqslant m_0$. Our main result states that, if $n_0 = m_0$, then the G-action is, up to a finite covering, an algebraic action. We use this to obtain a complete characterization of a large family of G-actions, thus providing a partial positive answer to the conjecture proposed in Zimmer's program for pseudoRiemannian manifolds. ***To cite this article: R. Quiroga-Barranco, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Géométrie pseudoRiemannienne et actions des groupes de Lie simples. Soit G un groupe de Lie simple non compact connexe agissant isométriquement sur une variété pseudoRiemannienne compacte connexe M. Dénotez avec n_0 et m_0 la dimension des sous-espaces nuls maximales tangents á G et M, respectivement. Alors nous avons toujours $n_0 \leqslant m_0$. Notre résultat principal déclare que, si $n_0 = m_0$, alors le action de G est, jusqu'à une revêtement finie, une action algébrique. Nous employons ceci pour obtenir une caractérisation complète d'une famille nombreuse de actions de G, de ce fait fournissant une réponse positive partielle à la conjecture proposé dans le programme de Zimmer pour le variété pseudoRiemannienne. ***Pour citer cet article : R. Quiroga-Barranco, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

1. Introduction

Let G be a connected noncompact simple Lie group acting on a connected compact manifold M preserving a finite smooth measure. Robert Zimmer formulated in [11] the problem of classifying such actions. Moreover, it was considered in [11] the problem of proving that any such G-action can be built out of algebraic G-actions. The latter are given by double cosets $K\backslash L/\Gamma$, where L is a Lie group into which there is a nontrivial homomorphism $G \to L$, Γ is a lattice and K is a compact subgroup that centralizes the image of G in L. The G-action is then given by left translations. The formulation of this problem is known as Zimmer's program. Following several key works (see [3–5,13]), the

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current research towards solving this problem considers actions of G that preserve some sort of geometric structure. The work of both Gromov and Zimmer shows that rigid structures in the sense of Gromov (see [4]) are particularly useful. Among such structures, the pseudoRiemannian metrics are one of the more natural to consider. In this Note we take this as our starting point to provide a contribution to Zimmer's program for actions preserving a geometric structure.

It is well known that G carries bi-invariant pseudoRiemannian metrics. Hence, if G acts on M preserving a pseudoRiemannian metric, we can compare the geometries of both spaces and use this to try to classify them. If we denote by n_0 and m_0 the dimension of maximal lightlike tangent subspaces of G and M, respectively, then two basic facts appear. The number n_0 is an invariant of G and $n_0 \leqslant m_0$ for any isometric G-action. In this Note we announce a result that completely characterizes those G-actions for which $n_0 = m_0$. This should be compared with a work of Bader and Nevo (see [1]) where a result similar in spirit is obtained for actions preserving a conformal pseudoRiemannian structure, though our methods are completely different.

2. Actions of simple Lie groups on pseudoRiemannian manifolds

Our first result is the following:

Theorem 2.1. *Let G be a connected noncompact simple Lie group. If G acts faithfully and topologically transitively on a compact manifold M preserving a pseudoRiemannian metric such that $n_0 = m_0$, then the G-action on M is ergodic and engaging, and there exist*:

(1) *a finite covering $\widehat{M} \to M$,*
(2) *a connected Lie group L that contains G as a factor,*
(3) *a cocompact discrete subgroup Γ of L and a compact subgroup K of $C_L(G)$,*

for which the G-action on M lifts to $\widehat{M}$ so that $\widehat{M}$ is G-equivariantly diffeomorphic to $K\backslash L/\Gamma$. Furthermore, there is an ergodic and engaging G-invariant finite smooth measure on L/Γ.

In our arguments below, we assume that G and M satisfies the hypotheses of Theorem 2.1. To prove this result we first obtain an isometric splitting of a covering of M. The first step towards such splitting is given by the following result.

Lemma 2.2. *The group G acts everywhere locally freely with nondegenerate orbits. The metric induced by M on the G-orbits is given by a bi-invariant pseudoRiemannian metric on G that does not depend on the G-orbit. Moreover, the normal bundle to the G-orbits is integrable.*

The proof of this result proceeds as follows. Everywhere local freeness is obtained from the results in [4] or [9] (see also [10]). This allows us to trivialize the tangent bundle to the G-orbits and by considering the action on its ergodic components we can prove that such G-orbits are nondegenerate as a consequence of the condition $n_0 = m_0$ together with the simplicity of G. Denote with $T\mathcal{O}^\perp$ the normal bundle to the G-orbits. Then it is easy to prove that $n_0 = m_0$ implies that $T\mathcal{O}^\perp$ is definite. A curvature operator is then introduced for the bundle $T\mathcal{O}^\perp$, which is the obstruction for the integrability of $T\mathcal{O}^\perp$. By following the proofs of Lemma 9.1 and Theorem 9.2 in [2] (or the arguments from [4]), we can obtain, at almost every point in M, a Lie algebra of local Killing fields isomorphic to $\mathfrak{g}$ (the Lie algebra of G) that vanish at the given point. Using this and the fact that $T\mathcal{O}^\perp$ is definite, we can prove the vanishing of the curvature operator and thus the integrability of $T\mathcal{O}^\perp$ follows. Then by considering a leaf N of the foliation induced by $T\mathcal{O}^\perp$ we prove the following result. It is a consequence of both the completeness of G and N.

Proposition 2.3. *The map $G \times N \to M$ obtained from the restriction of the G-action to N is an isometric covering map. Moreover, there is a discrete subgroup Γ_0 of $\mathrm{Iso}(G \times \widetilde{N})$ such that $(G \times \widetilde{N})/\Gamma_0 \to M$ is a finite covering.*

The next step is to investigate the structure of the isometry group of G with a bi-invariant metric. The relevant result is the following.

Proposition 2.4. *The isometry group* $\mathrm{Iso}(G)$ *has finitely many components and* $\mathrm{Iso}(G)_0 = L(G)R(G)$*, the group generated by the left and right translations. Moreover, for any connected complete Riemannian manifold* $\widetilde{N}$ *the group* $\mathrm{Iso}(G \times \widetilde{N})$ *has finitely many connected components and* $\mathrm{Iso}(G \times \widetilde{N})_0 = L(G)R(G) \times \mathrm{Iso}(\widetilde{N})_0$.

This is proved by studying the properties of G as a pseudoRiemannian symmetric space. It also requires the application of the de Rham–Wu decomposition theorem for pseudoRiemannian manifolds. Then we prove that the topological transitivity of the G-action on M is enough to show that Singer's Theorem (see [7]) can be applied to conclude that $\widetilde{N}$ is a homogeneous pseudoRiemannian manifold, say $\widetilde{N} = K\backslash H$, with H a connected Lie group and K a compact subgroup.

The above argument together with Propositions 2.3 and 2.4 provide a finite covering space $\widehat{M} = (G \times K\backslash H)/\Gamma$ of M, where Γ is a discrete subgroup of $L(G)R(G) \times H$. Furthermore, we can prove that the G-action lifts to $\widehat{M}$ and use this to prove that Γ is actually a subgroup of $R(G) \times H = G \times H$. By defining $L = G \times H$ we find that most of Theorem 2.1 has been proven.

To complete the proof of Theorem 2.1 it only remains to show that the G-actions on M and L/Γ are ergodic and engaging. For M this is achieved by studying the properties of the transverse (definite) pseudoRiemannian structure of the G-orbits and applying Molino's machinery. For L/Γ we apply similar techniques, but we actually have to take a further finite covering that replaces L, Γ and $\widehat{M}$ so that the ergodic and engagement conditions are satisfied.

We observe that Theorem 2.1 has no rank restrictions on G, but provides no precise information on the structure of the group L. For higher real rank groups we have the following result, which provides a complete description of the group L that occurs in Theorem 2.1 and so an important improvement towards Zimmer's program for actions preserving a geometric structure.

Theorem 2.5. *Let G be a connected noncompact simple Lie group with finite center and* $\mathrm{rank}_{\mathbb{R}}(G) \geqslant 2$. *If G acts faithfully and topologically transitively on a compact manifold M preserving a pseudoRiemannian metric such that* $n_0 = m_0$*, then there exist*:

(1) *a finite covering* $\widehat{M} \to M$,
(2) *a connected isotypic semisimple Lie group L with finite center that contains G as a factor,*
(3) *a cocompact irreducible lattice Γ of L and a compact subgroup K of* $C_L(G)$,

for which the G-action on M lifts to $\widehat{M}$ so that $\widehat{M}$ is G-equivariantly diffeomorphic to $K\backslash L/\Gamma$. Hence, up to fibrations with compact fibers, M is G-equivariantly diffeomorphic to $K\backslash L/\Gamma$ and L/Γ.

The proof of Theorem 2.5 builds on the conclusions of Theorem 2.1. By using the ergodicity obtained from 2.1 we are able to apply the main result in [8] to conclude that the G-action is essentially free on $\widehat{M}$ and that it is free on L/Γ. Given this, we then apply the main result in [12] to conclude that L is semisimple. With such arguments, and given the techniques of [12], we observe that Theorem 2.5 ultimately depends on Zimmer's cocycle superrigidity. Next, we apply the structure theory of semisimple Lie groups to show that the ergodicity of the G-action on L/Γ implies that Γ has a finite index subgroup Z of the center of H. By modding out by Z we can assume that L has finite center. Then we apply the structure theory of finite center semisimple Lie groups to prove that Γ is irreducible and thus L is isotypic.

3. A classification theorem for actions of simple Lie groups

Our final result proves that the condition $n_0 = m_0$ completely characterizes a large family of algebraic actions, thus providing a partial positive answer to the conjecture proposed in Zimmer's program for actions preserving a geometric structure.

Theorem 3.1. *Let G be a connected noncompact simple Lie group with finite center and* $\mathrm{rank}_{\mathbb{R}}(G) \geqslant 2$. *Assume that G acts faithfully on a compact manifold X. Then the following conditions are equivalent.*

(1) *There is a finite covering* $\widehat{X} \to X$ *for which the* G*-action on* X *lifts to a topologically transitive* G*-action on* $\widehat{X}$ *that preserves a pseudoRiemannian metric satisfying* $n_0 = m_0$.

(2) *There is a connected isotypic semisimple Lie group* L *with finite center that contains* G *as a factor, a cocompact irreducible lattice* Γ *of* L *and a compact subgroup* K *of* $C_L(G)$ *such that* $K\backslash L/\Gamma$ *is a finite covering of* X *with* G*-equivariant covering map.*

The proof relies on Theorem 2.5 for one direction of the equivalence, and an easy construction of a suitable metric on double cosets $K\backslash L/\Gamma$ as above for the other direction. A number of consequences can be obtained from our theorems. We can also extend our arguments to finite volume manifolds. Such results, with a detailed account of the proofs presented here, will appear in [6].

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