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Mathematical Analysis

# Abstract theory of universal series and an application to Dirichlet series ☆

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**Abstract**

We present an abstract theory of universal series; in particular, we give a necessary and sufficient condition for the existence of universal series of a certain type. Most of the known results can be proved or strengthened by using this condition. We also obtain new results, for example, related to universal Dirichlet series. ***To cite this article: V. Nestoridis, C. Papadimitropoulos, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***


**Résumé**

**Théorie abstraite des séries universelles et une application aux séries de Dirichlet.** Ainsi nous obtenons des démonstrations simples et des versions améliorées de la plupart de résultats connus. Nous obtenons aussi des résultats nouveaux, par example dans le cas de séries de Dirichlet. ***Pour citer cet article : V. Nestoridis, C. Papadimitropoulos, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***
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## Version française abrégée

Soit $X$ un espace vectoriel sur le corps $\mathbb{K} = \mathbb{C}$ ou $\mathbb{R}$, muni d'une métrique $\rho$ compatible avec les opérations $+, \cdot$ et invariante par translation. Soit $x_0, x_1, \ldots$ une suite fixée d'éléments de $X$. Une suite $a = (a_0, a_1, a_2, \ldots) \in \mathbb{K}^{\mathbb{N}_0}$ appartient à la classe $U$ si la suite $\sum_{j=0}^{n} a_j x_j$, $n = 0, 1, 2, \ldots,$ est dense dans $X$.

Soit $A$ un sous-espace vectoriel de $\mathbb{K}^{\mathbb{N}_0}$, muni d'une distance $d$ invariante par translation et telle que les opérations $+$ et $\cdot$ sont continues. On suppose que : (a) $(A, d)$ est complet. (b) Les projections $A \ni a \to a_m \in \mathbb{K}$ sont continues pour tout $m = 0, 1, 2, \ldots.$ (c) L'ensemble $G = \{a = (a_n)_{n=0}^{\infty} \in \mathbb{K}^{\mathbb{N}_0}\colon \{n\colon a_n \neq 0\}$ est fini$\}$ est contenu dans $A$. (d) $G$ est dense dans $A$.

On note $e_0 = (1, 0, 0, \ldots)$, $e_1 = (0, 1, 0, \ldots)$, $e_2 = (0, 0, 1, 0, \ldots), \ldots.$

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**Définition 0.1.** Une suite $a=(a_n)_{n=0}^{\infty}\in A$ appartient à la classe $U_A$, si pour tout $x\in X$, il existe une suite $\{\lambda_n\}_{n=1}^{\infty}$ dans $\mathbb{N}=\{0,1,2,\ldots\}$ strictement croissante, telle que

$$\text{(i)}\ \sum_{j=0}^{\lambda_n} a_j x_j \to x,\ \text{quand}\ n\to+\infty \quad \text{et} \quad \text{(ii)}\ \sum_{j=0}^{\lambda_n} a_j e_j \to a,\ \text{quand}\ n\to+\infty.$$

Évidemment $U_A\subset U\cap A$.

**Théorème 0.1.** *Sous les conditions précédentes, les assertions suivantes sont équivalentes* :

(1) $U_A\neq\emptyset$.
(2) *Pour tout* $p\in\mathbb{N}$, $x\in X$ *et* $\varepsilon>0$, *il existe* $M\geqslant p$ *et* $\beta_p,\beta_{p+1},\ldots,\beta_M$ *dans* $\mathbb{K}$, *tels que* $\rho(\sum_{j=p}^{M}\beta_j x_j,x)<\varepsilon$ *et* $d(\sum_{j=p}^{M}\beta_j e_j,0)<\varepsilon$.
(3) *Pour tout* $x\in X$ *et* $\varepsilon>0$, *il existe* $M\in\mathbb{N}$ *et* $\beta_0,\beta_1,\ldots,\beta_M$ *dans* $\mathbb{K}$, *tels que* $\rho(\sum_{j=0}^{M}\beta_j x_j,x)<\varepsilon$ *et* $d(\sum_{j=0}^{M}\beta_j e_j,0)<\varepsilon$.
(4) $U_A$ *est dense et* $G_\delta$ *dans* $A$.

## 1. Abstract theory

Let $X$ be a vector space over the field $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, endowed with a metric $\rho$ compatible with the operations $+,\cdot$ and invariant under translations. Let $x_0,x_1,x_2,\ldots$ be a fixed sequence of elements of $X$. A sequence $a=(a_0,a_1,a_2,\ldots)\in\mathbb{K}^{\mathbb{N}_0}$ belongs to the class $U$ if the sequence $\sum_{j=0}^{n}a_j x_j$, $n=0,1,2,\ldots,$ is dense in $X$. If $U\neq\emptyset$, then $U$ is automatically dense and $G_\delta$ in the space $\mathbb{K}^{\mathbb{N}_0}$, endowed with the Cartesian topology [6]. A necessary and sufficient condition so that $U\neq\emptyset$ is the following:

For every $p\in\mathbb{N}=\{0,1,2,\ldots\}$ the closure in $X$ of the linear span of $x_p,x_{p+1},\ldots$ is equal to $X$ [6].

Let $A$ be a vector subspace of $\mathbb{K}^{\mathbb{N}_0}$, endowed with a metric $d$ compatible with $+$ and $\cdot$ and invariant under translations. We assume that: (a) $(A,d)$ is complete; (b) the projections $A\ni a\to a_m\in\mathbb{K}$ are continuous for all $m=0,1,2,\ldots$; (c) the set $G=\{a=(a_n)_{n=0}^{\infty}\in\mathbb{K}^{\mathbb{N}_0}\colon \{n\colon a_n\neq 0\}$ is finite$\}$ is contained in $A$ and (d) $G$ is dense in $A$.

We denote $e_0=(1,0,0,\ldots)$, $e_1=(0,1,0,\ldots)$, $e_2=(0,0,1,0,\ldots),\ldots.$

**Definition 1.1.** A sequence $a=(a_n)_{n=0}^{\infty}\in A$ belongs to the class $U_A$, if for every $x\in X$, there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $\mathbb{N}$, such that

$$\text{(i)}\ \sum_{j=0}^{\lambda_n} a_j x_j \to x,\ \text{as}\ n\to+\infty \quad \text{and} \quad \text{(ii)}\ \sum_{j=0}^{\lambda_n} a_j e_j \to a,\ \text{as}\ n\to+\infty.$$

Without loss of generality, we can assume $\lambda_n<\lambda_{n+1}$ for all $n\in\mathbb{N}$, $n\geqslant 1$.

It is obvious that $U_A\subset U\cap A$. Furthermore, if $a\in U_A$ and $b\in G$, then $a+b\in U_A$; in particular, if $a=(a_0,a_1,a_2,\ldots,)\in U_A$ and $p\in\mathbb{N}$, then $(0,\ldots,0,a_p,a_{p+1},\ldots)\in U_A$.

**Remark 1.** In Definition 1.1 it is equivalent to realize approximation (ii) by a strictly increasing sequence $\{\mu_n\}_{n=1}^{\infty}$ in $\mathbb{N}$ independent of $x\in X$ and then realize approximation (i) by a subsequence $\{\mu_n'\}_{n=1}^{\infty}$ of $\{\mu_n\}_{n=1}^{\infty}$ depending on $x\in X$ (see [20]).

**Theorem 1.2.** *Under the previous assumptions the following are equivalent*

(1) $U_A\neq\emptyset$.
(2) *For every* $p\in\mathbb{N}$, $x\in X$ *and* $\varepsilon>0$, *there exist* $M\geqslant p$ *and* $\beta_p,\beta_{p+1},\ldots,\beta_M$ *in* $\mathbb{K}$, *such that* $\rho(\sum_{j=p}^{M}\beta_j x_j,x)<\varepsilon$ *and* $d(\sum_{j=p}^{M}\beta_j e_j,0)<\varepsilon$.

(3) *For every $x \in X$ and $\varepsilon > 0$, there exist $M \in \mathbb{N}$ and $\beta_0, \beta_1, \ldots, \beta_M$ such that $\rho(\sum_{j=0}^{M} \beta_j x_j, x) < \varepsilon$ and $d(\sum_{j=0}^{M} \beta_j e_j, 0) < \varepsilon$.*
(4) *$U_A$ is dense and $G_\delta$ in $A$.*

**Sketch of the proof.** (1) $\Rightarrow$ (2). Let $a = (a_0, a_1, a_2, \ldots) \in U_A$. Then there exists $p' \geqslant p$ so that $\beta = (0, 0, \ldots, 0, a_{p'}, a_{p'+1}, \ldots) \in U_A$ and $d(\beta, 0) < \varepsilon/2$. Since $\lambda_n < \lambda_{n+1}$, we can find $M \geqslant p'$ so that $\rho(\sum_{j=p'}^{M} a_j x_j, x) < \varepsilon/2$ and $d(\sum_{j=p'}^{M} a_j e_j, \beta) < \varepsilon/2$. By the triangle inequality we obtain $d(\sum_{j=p'}^{M} a_j e_j, 0) < \varepsilon$. We set $\beta_j = 0$ for $p \leqslant j < p'$ and $\beta_j = a_j$ for $p' \leqslant j \leqslant M$ and we are done.

(2) $\Rightarrow$ (3). It suffices to apply (2) with $p = 0$.

(3) $\Rightarrow$ (4). From (3) it follows easily that $X$ is separable; let $\varphi_j$, $j = 1, 2, \ldots,$ be a dense denumerable subset of $X$. For $n \in \mathbb{N}$ and $j, s \geqslant 1$ we consider the sets $E(n, j, s) = \{a \in A\colon\ \rho(\sum_{l=0}^{n} a_l x_l, \varphi_j) < 1/s\}$ and $\Xi(n, s) = \{a \in A\colon\ d(\sum_{l=0}^{n} a_l e_l, a) < 1/s\}$. One can easily see that $U_A = \bigcap_{j,s} \bigcup_{n=0}^{\infty} [E(n, j, s) \cap \Xi(n, s)]$ and that $E(n, j, s)$ and $\Xi(n, s)$ are open subsets of $A$. Therefore $U_A$ is $G_\delta$ in $A$. Since $A$ is complete, we can use Baire's theorem. It suffices to prove that $\bigcup_{n=0}^{\infty} [E(n, j, s) \cap \Xi(n, s)]$ is dense in $A$, for all $j$ and $s$. Let $b \in G$ and $\varepsilon > 0$. Since $G$ is dense in $A$, it suffices to find $a \in A$ and $n \in N$, such that $d(a, b) < \varepsilon$, $a \in E(n, j, s)$ and $a \in \Xi(n, s)$. We are looking for $a$ of the form $a = b + \gamma$, with $\gamma \in G$. By (3), there exists $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_M, 0, 0, \ldots) \in G$ so that $d(\gamma, 0) < \varepsilon$ and $\rho(\sum_{l=0}^{M} \gamma_l x_l, \varphi_j - \sum_{l=0}^{\infty} b_l x_l) < 1/s$. We set $a = b + \gamma$ and we choose $n \in \mathbb{N}$ after the supports of $b, \gamma, a$. Then, the proof is complete since $\sum_{l=0}^{n} a_l e_l = a$ and $d(\sum_{l=0}^{n} a_l e_l, a) = 0 < 1/s$.

(4) $\Rightarrow$ (1). This is obvious. □

**Remark 2.** If the conditions (1)–(4) of Theorem 1.2 are satisfied, then $U \cap A \neq \emptyset$, because $U \cap A \supset U_A$. One can easily see that, if $U \cap A \neq \emptyset$, then automatically $U \cap A$ is dense and $G_\delta$ in $A$.

**Remark 3.** A stronger condition than (d) is the following:

($d^1$) For every $a = (a_0, a_1, \ldots) \in A$ we have $\sum_{l=0}^{N} a_l e_l \to a$, as $N \to +\infty$.

If $A$ satisfies (a)–($d^1$), then $U_A = U \cap A$ and conditions (1)–(4) of Theorem 1.2 are equivalent to '$U \cap A \neq \emptyset$', which is also equivalent to '$U \cap A$ is dense and $G_\delta$ in $A$'.

**Corollary 1.3.** *Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a bijection. If condition* (1), *of Theorem* 1.2 *and conditions* (a)–(d) *are satisfied, then there exists $a \in A$ such that, for every $x \in X$, there exists a strictly increasing sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $\mathbb{N}$ such that $d(\sum_{j=0}^{\lambda_n} a_{\varphi(j)} e_{\varphi(j)}, a) \to 0$ and $\rho(\sum_{j=0}^{\lambda_n} a_{\varphi(j)} x_{\varphi(j)}, x) \to 0$, as $n \to +\infty$. The set of such $a$'s is dense and $G_\delta$ in $A$. The proof is based on condition* (3) *of Theorem* 1.2, *since the image by a bijection of a finite set is also finite.*

**Corollary 1.4.** *Let $T_n : A \to X$, $n = 0, 1, 2, \ldots,$ be a sequence of continuous functions such that, for every $g \in G$, there exists a strictly increasing sequence $\{n_\tau\}_{\tau=1}^{\infty}$ in $\mathbb{N}$ satisfying $T_{n_\tau}(g) \to \sum_{j=0}^{\infty} g_j x_j$ as $\tau \to +\infty$. We also assume that $A$ satisfies* (a)–(d) *and that* (1) *of Theorem* 1.2 *holds.*

*Then, there exists a sequence $a = (a_0, a_1, \ldots)$ in $A$, such that, for every $x \in X$, there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $\mathbb{N}$ with $d(\sum_{j=0}^{\lambda_n} a_j e_j, a) \to 0$ and $\rho(T_{\lambda_n}(a), x) \to 0$, as $n \to +\infty$. The set of such $a$'s is dense and $G_\delta$ in $A$.*

The proof extends results from [17,9]. We can apply Corollary 1.4 to the partial sums $T_n = \sum_{j=0}^{n} a_j x_j$ or averages of them [17,9]. We can also consider a metric space $J$ of parameters which is compact or hemicompact (that is, there exists an increasing sequence of compact sets $J_m$, $m = 1, 2, \ldots,$ such that every compact set $L \subset J$ is contained in some $J_m$). We consider continuous maps $T_n : J \times A \to X$, $n = 0, 1, 2, \ldots.$ We assume that, for $g \in G$, we have $\sup_{\xi \in L} \rho(T_{n_\tau}(\xi, g), \sum_{j=0}^{\infty} g_j x_j) \to 0$, as $\tau \to +\infty$, for every compact set $L \subset J$. We also assume conditions (a)–(d) and condition (1) of Theorem 1.2. Then, there exists $a \in A$, such that for every $x \in X$, there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $\mathbb{N}$ with $d(\sum_{j=0}^{\lambda_n} a_j e_j, a) \to 0$ and $\sup_{\xi \in L} \rho(T_{\lambda_n}(\xi, a), x) \to 0$, as $n \to +\infty$, for all $L \subset J$ compact.

The set of such $a$'s is dense and $G_\delta$ in $A$.

In the case of holomorphic functions we may also have an extra parameter, the center of expansion.

A first application gives results in the sense of [11,13]. Let $\Omega \neq \mathbb{C}$ be a simply connected domain and let $\zeta \in \Omega$. In $H(\Omega)$ we denote be $\tilde{d}$ a standard metric inducing the topology of uniform convergence on compacta. We set $A = \{a(f)\colon f \in H(\Omega)\}$ and $d(a(f), a(g)) = \tilde{d}(f, g)$, where $a(f) = \{\frac{f^{(n)}(\zeta)}{n!}\}_{n=0}^{\infty}$. Then assumptions (a)–(d) are valid. Furthermore let $K \subset \mathbb{C}$ be compact, $K \cap \Omega = \emptyset$ with $K^c$ connected. We set $X = A(K)$ and let $x_j$, $j = 0, 1, 2, \ldots,$ be the restriction of the function $(z-\zeta)^j$ on $K$. An application of Mergelyan's theorem implies that condition (3) of Theorem 1.2 is satisfied and the result follows. More generally applying the previous abstract theory of universal series, we can give simple proofs of most of the known results [1,4–19] and obtain new ones. In particular we describe an application to Dirichlet series, where we prove two new results (Theorems 2.8 and 2.9 below) related to a result of Bayart [3].

## 2. Dirichlet series

For $a = (a_1, a_2, \ldots) \in \mathbb{C}^{\mathbb{N}^*}$, where $\mathbb{N}^* = \{1, 2, \ldots\}$, and $\sigma \in \mathbb{R}$ we denote $\|a\|_\sigma = \sum_{n=1}^{\infty} |a_n| n^{-\sigma} \in [0, +\infty]$.

**Lemma 2.1** ([2] see also [3]). *Let $f$ be holomorphic in $\{z \in \mathbb{C}\colon 1/2 < \operatorname{Re} z < 1\}$ and $K \subset \{z \in \mathbb{C}\colon 1/2 < \operatorname{Re} z < 1\}$ compact. Let $\varepsilon > 0$ and $n_0 \in \mathbb{N}$. Then there exist $n_1 > n_0$ and complex numbers $(a_i)_{n_0+1}^{n_1}$ with $|a_i| \leqslant 1$, such that $\sup_{z\in K} |f(z) - \sum_{n=n_0+1}^{n_1} a_n n^{-z}| < \varepsilon$.*

**Definition 2.2** [3]. A compact set $K \subset \mathbb{C}$ is said admissible (for Dirichlet series) if $K^c$ is connected and if it can be written as $K = K_1 \cup \cdots \cup K_d$, where each $K_i$ is contained in a strip $S_i = \{z \in \mathbb{C}\colon a_i \leqslant \operatorname{Re} z \leqslant b_i\}$ with $b_i - a_i < 1/2$, the strips $S_i$ being disjoint.

In the case $f \equiv 0$, Lemma 2 in [3, p. 173] takes the following form.

**Lemma 2.3.** *Let $K \subset \{z \in \mathbb{C}\colon \operatorname{Re} z \leqslant 0\}$ be an admissible compact set. Let $g : K \to \mathbb{C}$ be continuous on $K$ and holomorphic in $K^0$. Let $\sigma > 0$ and $\varepsilon > 0$. Then, there exists a Dirichlet polynomial $h(z) = \sum_{n=1}^{M} b_n n^{-z}$ such that $\sup_{z\in K} |h(z) - g(z)| < \varepsilon$ and $\|b\|_\sigma < \varepsilon$, where $b = (b_1, \ldots, b_M, 0, 0, \ldots)$.*

By the change of variable $z = w + \tau$, Lemma 2.3 yields the following.

**Lemma 2.4.** *Let $K \subset \mathbb{C}$ be an admissible compact set. Let $g : K \to \mathbb{C}$ be continuous on $K$ and holomorphic in $K^0$ and let $\varepsilon > 0$ and $\sigma > \max_{z\in K}(\operatorname{Re} z)$. Then, there exists a Dirichlet polynomial $h(z) = \sum_{n=1}^{M} a_n n^{-z}$, such that $\sup_{z\in K} |h(z) - g(z)| < \varepsilon$ and $\|a\|_\sigma < \varepsilon$, where $a = (a_1, \ldots, a_M, 0, 0, \ldots)$.*

**Lemma 2.5.** *There exists a sequence $K_m \subset \{z \in \mathbb{C}\colon \operatorname{Re} z \leqslant 0\}$, $m = 1, 2, \ldots,$ of admissible compact sets, such that each admissible compact set $K \subset \{z \in \mathbb{C}\colon \operatorname{Re} z \leqslant 0\}$ is contained in some $K_m$.*

**Lemma 2.6.** *There exists a sequence $\widetilde{K}_m \subset \{z \in \mathbb{C}\colon \operatorname{Re} z < 0\}$, $m = 1, 2, \ldots,$ of admissible compact sets, such that each admissible compact set $K \subset \{z \in \mathbb{C}\colon \operatorname{Re} z < 0\}$ is contained in some $\widetilde{K}_m$.*

**Theorem 2.7** [3]. *There exists a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-z}$, where $\sum_{n=1}^{\infty} |a_n| n^{-\sigma} < +\infty$ for all $\sigma > 0$, such that for every admissible compact set $K \subset \{z \in \mathbb{C}\colon \operatorname{Re} z \leqslant 0\}$ and every $g \in A(K)$, there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $\mathbb{N}^*$, such that $\sum_{j=1}^{\lambda_n} a_j j^{-z}$ converges to $g(z)$ uniformly on $K$, as $n \to +\infty$. Furthermore, the set of such Dirichlet series is dense and $G_\delta$ in the space of Dirichlet series that are absolutely convergent in $\{z\colon \operatorname{Re} z > 0\}$.*

**Remark 4.** In the proof of Theorem 2.7, Lemma 2.3 implies that condition (3) of Theorem 1.2 is valid. We also notice that the partial sums in Theorem 2.7 may be replaced by averages of them or by $T_n(a)$ according to Corollary 1.4.

A stronger version of Theorem 2.7 is the following.

**Theorem 2.8.** *There exists a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-z}$ where $\sum_{n=1}^{\infty} |a_n| n^{-\sigma} < +\infty$, for all $\sigma > 0$, such that for every admissible compact set $K \subset \{z \in \mathbb{C}\colon \operatorname{Re} z \leqslant 0\}$ and every entire function $g$, there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in*

$\mathbb{N}^*$ *such that for every* $l = 0, 1, 2, \ldots$ $(\sum_{j=1}^{\lambda_n} a_j j^{-z})^{(l)} \to g^{(l)}(z)$ *uniformly on* $K$, *as* $n \to +\infty$. *Furthermore, the set of such Dirichlet series is dense and* $G_\delta$ *in the space of Dirichlet series absolutely convergent in* $\{z\colon \operatorname{Re} z > 0\}$.

In the proof, if $\varepsilon > 0$ is given and $K \subset \{z\colon \operatorname{Re} z \leqslant 0\}$ is an admissible compact set, then we first find $k_0 \in \mathbb{N}^*$ so that $\sum_{k=k_0+1}^{\infty} \frac{1}{2^k} < \frac{\varepsilon}{2}$. Next we consider an open set $\Omega \supset K$, such that $\overline{\Omega}$ is an admissible compact set and $\max_{z\in\overline{\Omega}}(\operatorname{Re} z) < 1/k_0$. Then we apply Lemma 2.4 with $\sigma = 1/k_0$, to obtain uniform convergence on $\overline{\Omega}$. Weierstrass theorem on the open set $\Omega$ implies the uniform convergence on $K$ of derivatives of any order. This yields the result.

An application of Lemma 2.4 with $\sigma = 0$, implies the following.

**Theorem 2.9.** *There exists* $a = (a_n)_{n=1}^{\infty}$ *in* $l^1$, *such that, for every admissible compact set* $K \subset \{z \in \mathbb{C}\colon \operatorname{Re} z < 0\}$ *and for every* $g \in A(K)$, *there exists a sequence* $\{\lambda_n\}_{n=1}^{\infty}$ *in* $\mathbb{N}^*$, *such that* $\sum_{k=1}^{\lambda_n} a_k k^{-z} \to g(z)$ *uniformly on* $K$, *as* $n \to +\infty$. *Furthermore, the set of such* $a$*'s is dense and* $G_\delta$ *in* $l^1$.

## Note

After this Note was submitted for publication we have been informed that during a seminar in Orsay, A. Mouze presented, among other results, an equivalent form of Theorem 2.9 with different methods; see [O. Demanze, A. Mouze, Growth of coefficients of universal Dirichlet series, submitted for publication].

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