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Numerical Analysis

A vector Hamilton–Jacobi formulation for the numerical simulation of Euler flows

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Abstract

A vector Hamilton–Jacobi formulation of the Euler equations for fluids is studied numerically. The long term objective is to find the sensitivity of a flow with respect to a parameter, which is solution of the linearized Euler equations with Dirac singularities in the initial conditions. A Hamilton–Jacobi formulation uses integral of the primitive variable so that Dirac singularities become shocks. It is shown here that there are vector Hamilton–Jacobi formulations for any vector conservation laws and that they can be simulated numerically with packages such as GO++ which we adapted to the vector case both on structured and unstructured meshes for this purpose. ***To cite this article: P. Hoch, O. Pironneau, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Une formulation de type Hamilton–Jacobi vectoriel pour la simulation numérique des écoulements d'Euler. On présente, à des fins numériques, une formulation de type Hamilton–Jacobi vectoriel pour les lois de conservations comme les équations d'Euler pour les fluides compressibles. L'application visée est le calcul des sensibilités des écoulements par rapport à un paramètre car il faut alors résoudre une équation d'Euler linéarisée avec des masses de Dirac dans les conditions initiales alors qu'avec la formulation Hamilton–Jacobi les masses de Dirac deviennent des discontinuités de chocs. On montre ici que toute loi de conservation vectorielle admet une représentation Hamilton–Jacobi vectorielle et que ces nouvelles équations peuvent être intégrées numériquement par les techniques du logiciel GO++ par exemple, qui a été adapté aux cas vectoriels en maillage structuré et non-structuré pour cet objectif. ***Pour citer cet article : P. Hoch, O. Pironneau, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

On considère la loi de conservation vectorielle (1). En une dimension d'espace ($d = 1$), il est facile en intégrant l'équation en x de trouver une formulation de Hamilton–Jacobi équivalente. Cette relation entre les équations d'Hamilton–Jacobi et les lois de conservation est connue [12] mais elle est exploitée en général dans l'autre sense [11].

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Pour faire de même en deux dimensions d'espace on peut remarquer que la loi (1) est une divergence de vecteur pour les variables (x,t). Toute divergence dérivant d'un rotationnel (si le domaine est simplement connexe) on peut alors obtenir la forme intégrale (5). Ensuite on élimine la fonction auxiliaire c et on obtient (7). L'équivalence et l'unicité, pour un jeux de conditions aux limites arbitraires, est montrée (9), tant que le problème en W est bien posé. A titre d'illustration on explicite (7) pour les équations de Saint-Venant (voir (9)).

Cette nouvelle formulation n'aurait pas d'intérêt particulier (sauf pour appliquer les méthodes de Galerkin-discontinues aux équations d'Euler (cf. Cockburn et Shu [7])) s'il n'y avait pas le problème du calcul des sensibilités par rapport à des paramètres par exemple dans les conditions initiales. En effet si une variation du paramètre entraîne une variation de choc alors la dérivée par rapport à ce paramètre aura une masse de Dirac (cf. Bardos et al. [2]) ce qui rend son approximation numérique très difficile, alors que cette singularité devient un choc dans la formulation Hamilton–Jacobi.

Les tests numériques portent sur quatre exemples : convection pure, propagation d'une vague en eaux peu profondes, comparaison avec une formulation Hamilton–Jacobi standard lorsque celle-ci existe et une solution des équations d'Euler compressibles en régime supersonique uniforme à l'infini autour d'un cylindre. Le premier cas a été intégré avec un schéma de Lax, les deux suivants avec un schémas d'ordre 1 et 2 de Tadmor et al. [6] et le dernier avec une adaptation au cas vectoriel de la bibliothèque `GO++`.

1. Integral formulation

Consider a conservation law for $W(x,t)\in\mathcal{R}^p$, with $\mathcal{F}:\mathcal{R}^p\to\mathcal{R}^{d\times p}$,

$$\partial_t W+\nabla\cdot\mathcal{F}(W)=0,\quad t\in(0,T),\qquad W(x,0)=W^0(x),\quad x\in\mathcal{R}^d. \tag{1}$$

In one dimension ($d=1$) (1) is equivalent to

$$\partial_t V+\mathcal{F}(\partial_x V)=0,\quad V=\int^x W(y,t)\,\mathrm{d}y. \tag{2}$$

To obtain a similar equation in any dimension, one observes that if there exists $V(x,t)\in\mathcal{R}^{d\times p}$ such that

$$\partial_t V+\mathcal{F}(\nabla\cdot V)=0,\qquad V(0)=V^0, \tag{3}$$

then $W=\nabla\cdot V$ is solution of (1), provided that $\nabla\cdot V^0=W^0$: simply take the divergence of (3). In two dimensions there is a constructive proof (and probably also in 3d using differential forms [14]):

$$\partial_t W+\partial_x F_1(W)+\partial_y F_2(W)=0\quad\Leftrightarrow\quad\nabla_{t,x,y}\cdot\begin{pmatrix}W^{\mathrm{T}}\\ F_1(W)^{\mathrm{T}}\\ F_2(W)^{\mathrm{T}}\end{pmatrix}=0. \tag{4}$$

With $a(x,t),b(x,t),c(x,t)\in\mathcal{R}^p$ it is

$$(\,W\quad F_1(W)\quad F_2(W)\,)=(\,\partial_x c-\partial_y b\quad\partial_y a-\partial_t c\quad\partial_t b-\partial_x a\,) \tag{5}$$

for which the last two are rewritten as

$$\begin{aligned}&\partial_t c+F_1(\partial_x c-\partial_y b)-\partial_y a=0,\\ &\partial_t b-F_2(\partial_x c-\partial_y b)-\partial_x a=0.\end{aligned} \tag{6}$$

The change of variable $c\to c-\partial_y\int^t a$, $b\to b-\partial_x\int^t a$, leaves $\partial_x c-\partial_y b$ unchanged and then with $V=(c,-b)\in\mathcal{R}^{p\times p}$ and $\mathcal{F}=(F_1,F_2):\mathcal{R}^p\to\mathcal{R}^{p\times p}$ the system is

$$\partial_t V+\mathcal{F}(\nabla\cdot V)=0. \tag{7}$$

Example 1. With the shallow water equations:

$$\partial_t\rho+\nabla\cdot v=0,\qquad\partial_t v+\nabla\cdot\left(\frac{v\otimes v}{\rho}+\frac{\rho^2}{2}\right)=0 \tag{8}$$

we are led to

$$\nabla \cdot V = W = \begin{pmatrix} \rho \\ v_1 \\ v_2 \end{pmatrix}, \qquad \mathcal{F}(W) = \begin{pmatrix} W_2 & W_3 \\ \frac{W_2^2}{W_1} + \frac{W_1^2}{2} & \frac{W_2 W_3}{W_1} \\ \frac{W_2 W_3}{W_1} & \frac{W_3^2}{W_1} + \frac{W_1^2}{2} \end{pmatrix}. \tag{9}$$

1.1. Equivalence and non-uniqueness in R^d

Naturally there are many ways to pass from a system of conservation laws to a H–J systems and this is just one way. Furthermore V is unique only once the initial conditions are chosen.

Assume that W, solution of (1) exists and is unique. Choose a $V^0 \in \mathcal{R}^{d\times p}$ satisfying $\nabla \cdot V^0 = W^0$, for example by solving for $\Phi : \mathcal{R}^d \to \mathcal{R}^p$:

$$-\Delta \Phi = W^0 \text{ and let } V^0 = \nabla \Phi \text{ then define } V \text{ by (3).} \tag{10}$$

Taking the divergence of (3) shows that $\nabla \cdot V = W$ at all times verifies:

$$\partial_t \nabla \cdot V + \nabla \cdot \mathcal{F}(\nabla \cdot V) = 0, \qquad \nabla \cdot V(0) = W^0. \tag{11}$$

This shows that V satisfying (3) and that $\nabla \cdot V = W$ always exists. As pointed out by Bourgain et al. [5] this choice may not be the best one and may not lead to the best regularity on V.

Conversely, little is known on the vector Hamilton–Jacobi systems (3) with initial condition $V = V^0$ (for scalar case see e.g. [3] for existence and uniqueness results and [8] for the convergence of approximation schemes). To our knowledge the only resort is to use the same but reverse procedure and associate to it a conservation law by taking its divergence [13].

In the scalar case it is known [12] that the gradient of a H–J equation satisfies a hyperbolic system of conservation laws (HSCL) and numerical solutions of the H–J equation shave been implemented that way [11] and instabilities have been observed [9] when the HSCL are weakly hyperbolic.

1.2. Entropy and boundary conditions

Obviously integral forms of the entropy conditions would be needed as well. It will be interesting to see the link with the notion of entropy; for scalar case see [10].

For boundary conditions, note how difficult it is to translate those coming from HSCL (1) to the Hamilton–Jacobi vectorial formulation (3).

The case considered here is that of an object C with $W = W_d$ constant. Then one can extend W^0 inside C to be W_d and solve the problem in $\mathcal{R}^d$ instead of $\mathcal{R}^d \backslash C$ with $\mathcal{F} \approx 0$ in C.

2. Application to sensitivity analysis

Assume that the initial conditions W^0 of (1) depend upon a parameter a, then the sensitivity $V' = \partial_a V$ satisfies

$$\partial_t V' + \mathcal{F}'_W(W)\nabla \cdot V' = 0, \qquad V'(0) = V^{0\prime} \tag{12}$$

with the condition that $\nabla \cdot V^{0\prime} = W^{0\prime}$. In practice one can find such a $V^{0\prime}$ as in (10). Therefore when $W^{0\prime}$ has Dirac singularities due to the change of the shock positions in the initial conditions for instance (see Bardos et al. [2]) then $V^{0\prime}$ has only jump discontinuities. It is hopeless to expect a numerical scheme for (1) to work with Dirac masses while many schemes do work on (7) with jump discontinuities. Such numerical applications will be given in a subsequent publication. Here we propose to validate the method on several equations related to Euler flows because the answers are familiar.

3. Numerical examples

3.1. Passive convection

For a given velocity field u let us work on a bounded rectangular domain and call $\Sigma = \{x, t)$: $u(x, t) \cdot n < 0$, $x \in \Gamma\}$. Then consider

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad \rho(0) = \rho^0, \quad \rho = \rho_\Gamma \quad \text{on } \Sigma. \tag{13}$$

For this problem (7) is

$$\partial_t \vec{V} + \mathcal{F}(\rho) = 0, \quad \nabla \cdot \vec{V} = \rho, \quad \mathcal{F}(\rho) = \rho \vec{u}. \tag{14}$$

When Σ is the line $x_1 = 0$ and $u = (1, 0)^{\mathrm{T}}$ then ρ is extended to ρ_Σ whenever $x < 0$ and so $\partial_{x_1} V_1 = 0$. This implies that $\partial_{x_2} V_2(0, x_2, t) = \rho_\Sigma$. For a flow dominant in the direction $x_1 > 0$, an upwind finite difference explicit scheme is then

$$\vec{V}_{i,j}^{m+1} = \vec{V}_{i,j}^{m} - \frac{\delta t}{h}\big(V_1{}^m_{i,j} - V_1{}^m_{i-1,j} + V_2{}^m_{i,j+1} - V_2{}^m_{i,j+1}\big)\vec{u}_{i,j}. \tag{15}$$

Two results are shown on Fig. 1. Of course, this scheme is rather diffusive and would not work for any u.

3.2. Shallow waters

Next we consider the shallow water equations with zero velocity and constant depth at time zero except in a small region in the center where the water depth is doubled. Fig. 2 shows the result obtained by integrating (7), (9) with the first order scheme of Tadmor [6].

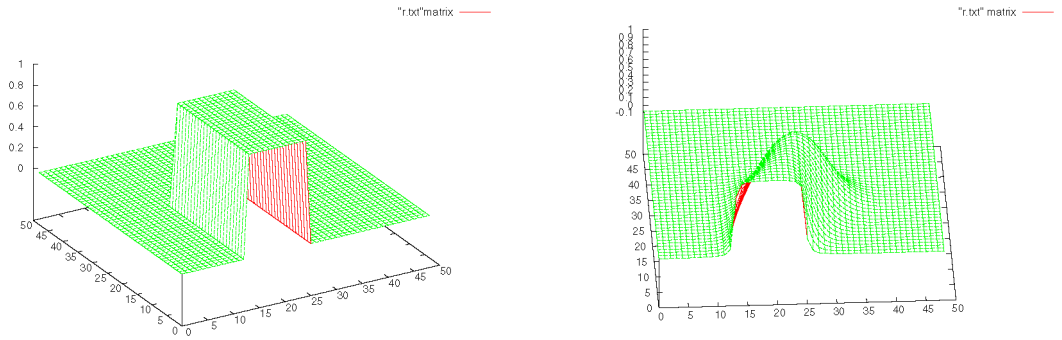

Fig. 1. Pure convection (see (13)) with velocity $u = (1, 0)$ (left) and $u = (1, 0.2)$ (right) with the order one scheme (15).

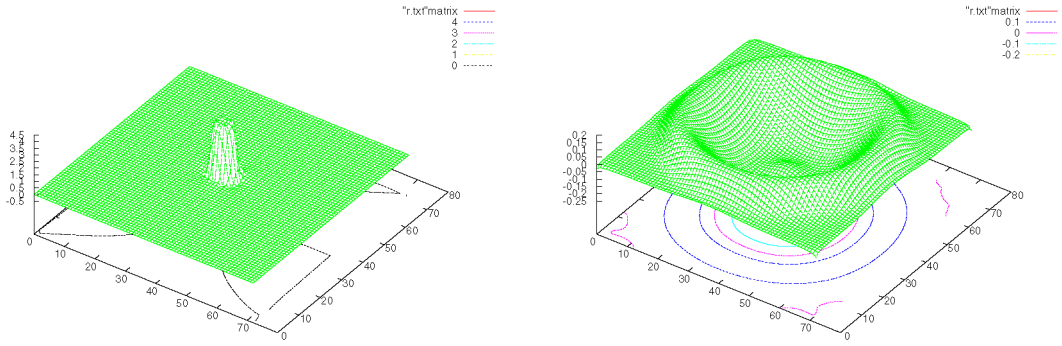

Fig. 2. Water wave at time 0 (left) and after 200 time steps (right) obtained with Tadmor's scheme of order 1.

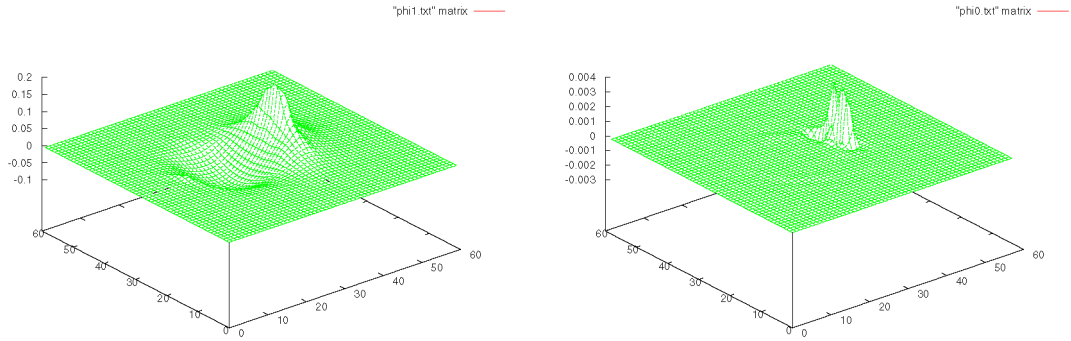

Fig. 3. Solution of (16) obtained with Tadmor's scheme of order two for the full vector Hamilton–Jacobi equation (left). Comparison with the scalar Hamilton–Jacobi formulation integrated with the same numerical scheme; the figure on the right displays the difference between the two computed solutions.

3.3. Comparison with the standard Hamilton–Jacobi when available

Consider the following example

$$\partial_t\phi + \frac{1}{2}\big(\partial_x\phi^2 + \partial_y\phi^2\big) = 0, \qquad \phi(0,x) = y\exp 40\big(x^2+y^2\big) \tag{16}$$

which, written in term of the velocity $u = \nabla\phi$ is

$$\partial_t\vec{u} + \nabla\cdot\mathcal{F}(\vec{u}) = 0 \quad \text{with } \mathcal{F}(\vec{u}) = \frac{1}{2}\begin{pmatrix} |u|^2 & 0 \\ 0 & |u|^2 \end{pmatrix}. \tag{17}$$

Accordingly V satisfies

$$\begin{aligned}
&\partial_t V + \mathcal{F}(\nabla\cdot V) = 0, \quad \text{i.e. with } u = \partial_x V_{11} + \partial_y V_{12},\ v = \partial_x V_{21} + \partial_y V_{22},\\
&\partial_t V_{11} + \frac{1}{2}\big(u^2+v^2\big), \qquad \partial_t V_{12} = 0,\\
&\partial_t V_{21} = 0, \qquad \partial_t V_{22} + \frac{1}{2}\big(u^2+v^2\big) = 0,\\
&V_{11} = V_{22} = y\exp 40\big(x^2+y^2\big), \qquad V_{12} = V_{21} = 0
\end{aligned} \tag{18}$$

and Eq. (16) is recovered. Comparison between the two formulations is shown on Fig. 3. The results are the same of course but the computing time is 4 time longer for the first case.

3.4. Euler flow

Finally we take the case of a compressible inviscid flow around a cylinder C. The domain is not simply connected but the flow is symmetric. The solution is stationary with inflow at Mach 4 implemented by translation so as to have a vanishing V at infinity and $W = W_d$ artificially maintained at each time step inside C. We extend to the vectorial case the ϵ-monotone schemes cf. [1] to our purpose.

$$\mathcal{F}^{LFX}\big(\overline{P}^{T_{i,1}}, \overline{P}^{T_{i,2}}, \ldots, \overline{P}^{T_{i,n\sup(i)}}\big) = \mathcal{F}\big(\widetilde{P}_i\big) - \nu(1-\kappa\times\rho^{\epsilon})\sum_{j=1}^{n\sup(i)} \beta_{j+1/2}\frac{\overline{P}^{T_{i,j}} + \overline{P}^{T_{i,j+1}}}{2}\cdot n^{j+1/2}. \tag{19}$$

Here $\overline{P}^{T_{i,j}}$ are the (constant) P1 divergence approximating (7) in the jth local element attached to the global node i. The divergence at node i, is computed by $\widetilde{P}_i = (\sum_{j=1}^{n\sup(i)}\theta_j\overline{P}^{T_{i,j}})/2\pi$, where θ_j is the angle defined by the ith-node and the two edges issued from it in the jth local element. $\beta_{j+1/2} = \tan(\frac{\theta_j}{2}) + \tan(\frac{\theta_{j+1}}{2})$, $\nu = \frac{C(L)}{\pi}$, $C(L)$ is the Lipschitz constant of $\mathcal{F}$, ρ is the lowest radius of inscribed circle issued from i, c is a real constant $\kappa\in[0,1]$ and $\epsilon = \frac{\log(1-\rho)}{\log(\rho)}$.

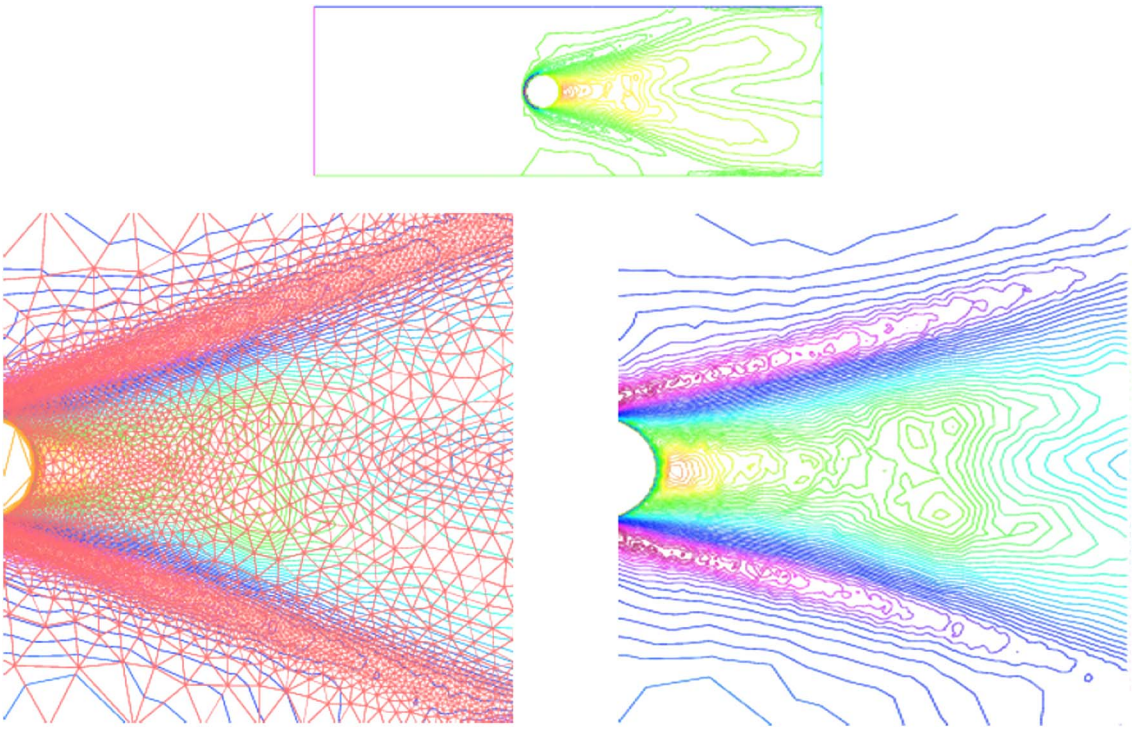

Fig. 4. (Stationary) density for a Mach 4 inflow.

To discretize in time, we use a second order Runge–Kutta method. And finally, to perform numerical simulation, we extend the GO++ library [4] in two ways: to the vectorial case and to unstructured triangular meshes. Results are shown on Fig. 4.

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