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Probability Theory

A curvature-dimension condition for metric measure spaces

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Abstract

We present a curvature-dimension condition $CD(K,N)$ for metric measure spaces (M,d,m). In some sense, it will be the geometric counterpart to the Bakry–Émery [D. Bakry, M. Émery, Diffusions hypercontractives, in: Séminaire de Probabilités XIX, in: Lecture Notes in Math., vol. 1123, Springer, Berlin, 1985, pp. 177–206. [1]] condition for Dirichlet forms. For Riemannian manifolds, it holds if and only if $\dim(M)\leqslant N$ and $\mathrm{Ric}_M(\xi,\xi)\geqslant K\cdot|\xi|^2$ for all $\xi\in TM$. The curvature bound introduced in [J. Lott, C. Villani, Ricci curvature for metric-measure spaces via optimal transport, Annals of Math., in press. [4]; K.T. Sturm, Generalized Ricci bounds and convergence of metric measure spaces, C. R. Acad. Sci. Paris, Ser. I 340 (2005) 235–238. [6]; K.T. Sturm, On the geometry of metric measure spaces. I, Acta Math., in press. [7]] is the limit case $CD(K,\infty)$.

Our curvature-dimension condition is stable under convergence. Furthermore, it entails various geometric consequences e.g. the Bishop–Gromov theorem and the Bonnet–Myers theorem. In both cases, we obtain the sharp estimates known from the Riemannian case. ***To cite this article: K.-T. Sturm, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Une condition de type courbure-dimension pour des espaces métriques mesurés. Nous présentons une condition de type courbure-dimension $CD(K,N)$ pour des espaces métriques mesurés (M,d,m), qui peut être considérée comme une contrepartie géométrique de celle de Bakry–Émery [D. Bakry, M. Émery, Diffusions hypercontractives, in: Séminaire de Probabilités XIX, in: Lecture Notes in Math., vol. 1123, Springer, Berlin, 1985, pp. 177–206. [1]] pour les formes de Dirichlet. Pour les variétés riemanniennes, elle est satisfaite si et seulement si $\dim(M)\leqslant N$ et $\mathrm{Ric}_M(\xi,\xi)\geqslant K\cdot|\xi|^2$ pour tout $\xi\in TM$. La borne de la courbure [J. Lott, C. Villani, Ricci curvature for metric-measure spaces via optimal transport, Annals of Math., in press. [4] ; K.T. Sturm, Generalized Ricci bounds and convergence of metric measure spaces, C. R. Acad. Sci. Paris, Ser. I 340 (2005) 235–238. [6] ; K.T. Sturm, On the geometry of metric measure spaces. I, Acta Math., in press. [7]] est le cas limite $CD(K,\infty)$.

Notre condition est stable pour la convergence. Elle comporte des conséquences géométriques diverses, comme les théorèmes de Bishop–Gromov et de Bonnet–Myers. Dans les deux cas, on obtient des estimations optimales connues dans le cas riemannien. ***Pour citer cet article : K.-T. Sturm, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

A *metric measure space* will always be a triple (M,d,m) where (M,d) is a complete separable metric space and m is a locally finite measure on M equipped with its Borel σ-algebra. The case $m(M)=0$ will be excluded. $\mathcal{P}_2(M,\mathsf{d})$

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denotes the L_2-Wasserstein space of probability measures on M and d_W the corresponding L_2-Wasserstein distance. The subspace of m-absolutely continuous measures is denoted by $\mathcal{P}_2(M,\mathsf{d},m)$.

Given a metric measure space (M,d,m) and a number $N\in\mathbb{R}$, $N\geqslant 1$ we define the *Rényi entropy functional* $S_N(\cdot|m):\mathcal{P}_2(M,\mathsf{d})\to\mathbb{R}$ with respect to m by

$$S_N(\nu|m):=-\int\rho^{-1/N}\,\mathrm{d}\nu$$

where ρ denotes the density of the absolutely continuous part ν^c in the Lebesgue decomposition $\nu=\nu^c+\nu^s=\rho m+\nu^s$ of $\nu\in\mathcal{P}_2(M,\mathsf{d})$. Note that $S_1(\nu|m)=-m(\mathrm{supp}[\nu^c])$. The functional $\widetilde{S}_N:=N+NS_N$ shares various properties with the relative Shannon entropy $\mathrm{Ent}(\cdot|m)$. For instance, if m is a probability measure then $\widetilde{S}_N(\cdot|m)\geqslant 0$ on $\mathcal{P}_2(M,\mathsf{d})$ and $\widetilde{S}_N(\nu|m)=0$ if and only if $\nu=m$. If $m(M)$ is finite then $\mathrm{Ent}(\nu|m)=\lim_{N\to\infty}N(1+S_N(\nu|m))$ for each $\nu\in\mathcal{P}_2(M,\mathsf{d})$.

Definition 1. Given two numbers $K,N\in\mathbb{R}$ with $N\geqslant 1$ we say that a metric measure space (M,d,m) satisfies the *curvature-dimension condition* $CD(K,N)$ iff for each pair $\nu_0,\nu_1\in\mathcal{P}_2(M,\mathsf{d},m)$ there exist an optimal coupling q of ν_0,ν_1 and a geodesic $\Gamma:[0,1]\to\mathcal{P}_2(M,\mathsf{d},m)$ connecting ν_0,ν_1 with

$$S_{N'}\big(\Gamma(t)|m\big)\leqslant-\int_{M\times M}\Big[\tau_{K,N'}^{(1-t)}\big(\mathsf{d}(x_0,x_1)\big)\cdot\rho_0^{-1/N'}(x_0)+\tau_{K,N'}^{(t)}\big(\mathsf{d}(x_0,x_1)\big)\cdot\rho_1^{-1/N'}(x_1)\Big]\,\mathrm{d}q(x_0,x_1)$$

for all $t\in[0,1]$ and all $N'\geqslant N$. Here ρ_i denotes the density of the absolutely continuous part of ν_i w.r.t. m (for $i=0,1$) and for each $\theta\in\mathbb{R}_+$

$$\tau_{K,N}^{(t)}(\theta):=\begin{cases}\infty, & \text{if } K\theta^2\geqslant(N-1)\pi^2,\\ t^{1/N}\Big(\sin\Big(\sqrt{\frac{K}{N-1}}\,t\theta\Big)\Big/\sin\Big(\sqrt{\frac{K}{N-1}}\,\theta\Big)\Big)^{1-1/N}, & \text{if } 0<K\theta^2<(N-1)\pi^2,\\ t, & \text{if } K\theta^2=0 \text{ or}\\ & \text{if } K\theta^2<0 \text{ and } N=1,\\ t^{1/N}\Big(\sinh\Big(\sqrt{\frac{-K}{N-1}}\,t\theta\Big)\Big/\sinh\Big(\sqrt{\frac{-K}{N-1}}\,\theta\Big)\Big)^{1-1/N}, & \text{if } K\theta^2<0 \text{ and } N>1.\end{cases}$$

Theorem 2. *Let M be a complete Riemannian manifold with Riemannian distance d and Riemannian volume m and let numbers $K,N\in\mathbb{R}$ with $N\geqslant 1$ be given.*

(i) *The metric measure space (M,d,m) satisfies the curvature-dimension condition $CD(K,N)$ if and only if the Riemannian manifold M has Ricci curvature $\geqslant K$ and dimension $\leqslant N$.*
(ii) *Moreover, in this case for every measurable function $V:M\to\mathbb{R}$ the weighted space $(M,\mathsf{d},V\,m)$ satisfies the curvature-dimension condition $CD(K+K',N+N')$ provided*

$$\mathrm{Hess}\,V^{1/N'}\leqslant-\frac{K'}{N'}\cdot V^{1/N'}$$

for some numbers $K'\in\mathbb{R}$, $N'>0$ in the following sense:

$$V(\gamma_t)^{1/N'}\geqslant\sigma_{K',N'}^{(1-t)}\big(\mathsf{d}(\gamma_0,\gamma_1)\big)V(\gamma_0)^{1/N'}+\sigma_{K',N'}^{(t)}\big(\mathsf{d}(\gamma_0,\gamma_1)\big)V(\gamma_1)^{1/N'}$$

for each geodesic $\gamma:[0,1]\to M$ and each $t\in[0,1]$. Here $\sigma_{K',N'}^{(t)}(\theta):=t^{-1/N'}\cdot\tau_{K',N'+1}^{(t)}(\theta)^{1+1/N'}$.

This essentially follows from estimates for the Jacobian of transport maps in [3] and [5]. The particular case of the $CD(0,N)$ condition has already been treated in [5] and later in [4]. In this particular case $K=0$, the assertions of the following Theorems 4 and 8 have been already deduced in [4]. In the borderline case $N=\infty$, the curvature-dimension condition $CD(K,N)$ reduces to the generalized lower Ricci curvature bound introduced independently in [4] and [6,7]. Also in this case, a stability result analogous to Theorem 8 from below has been derived in the above mentioned papers.

Let us have a closer look on the previous results if M is a subset of the real line equipped with the usual distance d and the 1-dimensional Lebesgue measure m.

Example 1.

(i) For each pair of real numbers $K > 0, N > 1$ the space $([0, L], \mathsf{d}, Vm)$ with $L := \sqrt{\frac{N-1}{K}}\pi$ and $V(x) = \sin(\sqrt{\frac{K}{N-1}}x)^{N-1}$ satisfies the curvature-dimension condition $CD(K, N)$.

(ii) For each pair of real numbers $K \leqslant 0$, $N > 1$ the space $(\mathbb{R}_+, \mathsf{d}, Vm)$ with $V(x) = \sinh(\sqrt{\frac{-K}{N-1}}x)^{N-1}$, if $K < 0$, and $V(x) = x^{N-1}$, if $K = 0$, satisfies the curvature-dimension condition $CD(K, N)$.

(iii) For each pair of real numbers $K < 0$, $N > 1$ the space $(\mathbb{R}, \mathsf{d}, Vm)$ with $V(x) = \cosh(\sqrt{\frac{-K}{N-1}}x)^{N-1}$ satisfies the curvature-dimension condition $CD(K, N)$.

Note that for $N \to \infty$ the weight V from example (iii) from above converges to the weight $V(x) = \exp(\frac{-K}{2}x^2)$. Also note that according to [2], the examples (i)–(iii) equipped with natural weighted Laplacians are also the prototypes for the Bakry–Émery curvature-dimension condition.

Proposition 3 (Generalized Brunn–Minkowski Inequality). *Assume that the metric measure space (M, d, m) satisfies the curvature-dimension condition $CD(K, N)$ for some real numbers $K, N \in \mathbb{R}$, $N \geqslant 1$. Then for all measurable sets $A_0, A_1 \subset M$ with $m(A_0) \cdot m(A_1) > 0$, all $t \in [0, 1]$ and all $N' \geqslant N$*

$$m(A_t)^{1/N'} \geqslant \tau_{K,N'}^{(1-t)}(\Theta) \cdot m(A_0)^{1/N'} + \tau_{K,N'}^{(t)}(\Theta) \cdot m(A_1)^{1/N'}$$

where A_t denotes the set of points γ_t on geodesics with endpoints $\gamma_0 \in A_0$ and $\gamma_1 \in A_1$ and where $\Theta = \inf_{x_0 \in A_0,\, x_1 \in A_1} \mathsf{d}(x_0, x_1)$ if $K \geqslant 0$ and $\Theta = \sup_{x_0 \in A_0,\, x_1 \in A_1} \mathsf{d}(x_0, x_1)$ if $K < 0$. In particular, if $K \geqslant 0$ then

$$m(A_t)^{1/N'} \geqslant (1-t) \cdot m(A_0)^{1/N'} + t \cdot m(A_1)^{1/N'}.$$

Now let us fix a point $x_0 \in \operatorname{supp}[m]$ and study the growth of the volume of concentric balls as well as the growth of the volume of the corresponding spheres:

$$v(r) := m\big(\overline{B}_r(x_0)\big), \qquad s(r) := \limsup_{\delta \to 0} \frac{1}{\delta} \cdot m\big(\overline{B}_{r+\delta}(x_0) \setminus B_r(x_0)\big).$$

Theorem 4 (Generalized Bishop–Gromov Volume Growth Inequality). *Assume that the metric measure space (M, d, m) satisfies the curvature-dimension condition $CD(K, N)$ for some $K, N \in \mathbb{R}$, $N \geqslant 1$. Then each bounded set $M' \subset M$ has finite volume. Moreover, either m is supported by one point or all points and all spheres have mass 0.*

More precisely, if $N > 1$ then for each fixed $x_0 \in \operatorname{supp}[m]$ and all $0 < r < R \leqslant \sqrt{\frac{N-1}{K \vee 0}} \cdot \pi$

$$\frac{s(r)}{s(R)} \geqslant \left(\frac{\sin(\sqrt{K/(N-1)}r)}{\sin(\sqrt{K/(N-1)}R)}\right)^{N-1} \quad \textit{and} \quad \frac{v(r)}{v(R)} \geqslant \frac{\int_0^r \sin(\sqrt{K/(N-1)}\,t)^{(N-1)}\,\mathrm{d}t}{\int_0^R \sin(\sqrt{K/(N-1)}\,t)^{N-1}\,\mathrm{d}t}$$

with $s(\cdot)$ and $v(\cdot)$ defined as above and with the usual interpretation of the RHS if $K \leqslant 0$. In particular, if $K = 0$

$$\frac{s(r)}{s(R)} \geqslant \left(\frac{r}{R}\right)^{N-1} \quad \textit{and} \quad \frac{v(r)}{v(R)} \geqslant \left(\frac{r}{R}\right)^{N}.$$

The latter also holds true if $N = 1$ and $K \leqslant 0$.

For each K and each *integer* $N > 1$ the simply connected spaces of dimension N and constant curvature $K/(N-1)$ provide examples where these volume growth estimates are sharp. But also for arbitrary *real* numbers $N > 1$ these estimates are sharp as demonstrated by Example 1 (i) and (ii) where equality is attained.

Corollary 5 (Doubling). *For each metric measure space (M, d, m) which satisfies the curvature-dimension condition $CD(K, N)$ for some $K, N \in \mathbb{R}$, $N \geqslant 1$, the doubling property holds on each bounded subset $M' \subset \operatorname{supp}[m]$. In particular, each bounded closed subset $M' \subset \operatorname{supp}[m]$ is compact.*

If $K \geqslant 0$ or $N = 1$ the doubling constant is $\leqslant 2^N$. Otherwise, it can be estimated by $2^N \cdot \cosh(\sqrt{\frac{-K}{N-1}}\,L)^{N-1}$ where L is the diameter of M'.

Corollary 6 (Hausdorff Dimension). *Each metric measure space (M, d, m) which satisfies the curvature-dimension condition CD(K, N) for some $K, N \in \mathbb{R}$, $N \geqslant 1$, has Hausdorff dimension $\leqslant N$.*

Corollary 7 (Generalized Bonnet–Myers Theorem). *For every metric measure space (M, d, m) which satisfies the curvature-dimension condition CD(K, N) for some real numbers $K > 0$ and $N \geqslant 1$ the support of m is compact and has diameter*

$$L \leqslant \sqrt{\frac{N-1}{K}}\,\pi.$$

In particular, if $K > 0$ and $N = 1$ then $\mathrm{supp}[m]$ *consists of one point.*

Theorem 8 (Stability under Convergence). *Let $((M_n, \mathsf{d}_n, m_n))_{n\in\mathbb{N}}$ be a sequence of normalized metric measure spaces where for each $n \in \mathbb{N}$ the space (M_n, d_n, m_n) satisfies the curvature-dimension condition CD(K_n, N_n) and has diameter $\leqslant L_n$. Assume that for $n \to \infty$*

$$(M_n, \mathsf{d}_n, m_n) \xrightarrow{\mathbb{D}} (M, \mathsf{d}, m)$$

and $(K_n, N_n, L_n) \to (K, N, L)$ for some triple $(K, N, L) \in \mathbb{R}^2$ satisfying $K \cdot L^2 < (N-1)\pi^2$. Then the space (M, d, m) satisfies the curvature-dimension condition CD(K, N) and has diameter $\leqslant L$.

Corollary 9 (Compactness). *For each triple $(K, L, N) \in \mathbb{R}^3$ with $K \cdot L^2 < (N-1)\pi^2$ the family $\mathbb{X}_1(K, N, L)$ of isomorphism classes of normalized metric measure spaces which satisfy the curvature-dimension condition CD(K, N) and which have diameter $\leqslant L$ is compact w.r.t. $\mathbb{D}$.*

For detailed proofs and further results see [8].

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