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Geometry/Algebra

Triangular hyperbolic buildings

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Abstract

We construct triangular hyperbolic polyhedra whose links are generalized 4-gons. The universal cover of such a polyhedron is a hyperbolic building, whose apartments are hyperbolic planes tessellated by regular triangles with angles $\pi/4$. The fundamental groups of the polyhedra are hyperbolic, torsion free, with property (T). ***To cite this article: R. Kangaslampi, A. Vdovina, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Immeubles hyperboliques triangulaires. On construit des polyèdres hyperboliques dont les links en chaque sommet sont des 4-gones généralizées. Leurs revêtements universels sont des immeubles dont les appartements sont des plans hyperboliques pavés par des triangles réguliers d'angles $\pi/4$. Les groupes fondamentaux de nos polyédres sont hyperboliques, sans torsion et ont la propriété (T). ***Pour citer cet article : R. Kangaslampi, A. Vdovina, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

Hyperbolic torsion free groups with property (T) have uncountably many nonisomorphic quotient groups $(\Gamma_\alpha)_{\alpha\in I}$ which are simple and with infinitely many conjugacy classes (see [8,10,11]). Such groups exist: the random group of Gromov [9], cocompact lattices of $\mathrm{Sp}(1,n)$ etc.

We give new examples of groups of this kind which are explicitly presented by generators and relations.

A *polyhedron* is a two-dimensional complex which is obtained from several oriented p-gons by identification of corresponding sides. Let us take a sphere of a small radius at a point of the polyhedron. The intersection of the sphere with the polyhedron is a graph, which is called the *link* at this point.

In this Note we construct polyhedra whose links at vertices are generalized 4-gons and whose faces are regular hyperbolic triangles with angles $\pi/4$. The universal covering of such a polyhedron is a hyperbolic building, see [6]. Moreover, with the metric introduced in [1, p. 165] it is a complete metric space of non-positive curvature in the sense of Alexandrov and Busemann [7]. It follows from [2] that the fundamental groups of our polyhedra satisfy the property

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(T) of Kazhdan. (Another relevant reference is [15].) So, our groups, which are explicitly presented by generators and relations, are hyperbolic, torsion free and they have property (T).

Definition 1.1. Let $\mathcal{P}(p,m)$ be a tessellation of the hyperbolic plane by regular polygons with p sides, with angles π/m at each vertex where m is an integer. A *hyperbolic building* is a polygonal complex X, which can be expressed as the union of subcomplexes called apartments such that:

1. Every apartment is isomorphic to $\mathcal{P}(p,m)$.
2. For any two polygons of X, there is an apartment containing both of them.
3. For any two apartments $A_1, A_2 \in X$ containing the same polygon, there exists an isomorphism $A_1 \to A_2$ fixing $A_1 \cap A_2$.

Our construction gives new examples of hyperbolic triangular buildings with regular triangles as chambers. Examples of hyperbolic buildings with right-angled triangles were constructed by Bourdon in [3]. His construction has been generalized by Świątkowski in [12].

2. Polygonal presentation and construction of polyhedra

Recall that a *generalized m-gon* is a connected, bipartite graph of diameter m and girth $2m$, in which each vertex lies on at least two edges. A graph is *bipartite* if its set of vertices can be partitioned into two disjoint subsets such that no two vertices in the same subset lie on a common edge. The vertices of one subset we will call black vertices, denoted by x_i, and the vertices of the other subset the white ones, denoted by y_i, $i \in \mathbb{Z}_+$. The *diameter* is the maximum distance between two vertices and the *girth* is the length of a shortest circuit.

We recall also the definition of a polygonal presentation introduced in [14]:

Definition 2.1. Suppose we have n disjoint connected bipartite graphs $G_1, G_2, \ldots, G_n$. Let P_i and Q_i be the sets of black and white vertices respectively in G_i, $i = 1, \ldots, n$; let $P = \bigcup P_i$, $Q = \bigcup Q_i$, $P_i \cap P_j = \emptyset$, $Q_i \cap Q_j = \emptyset$ for $i \neq j$ and let λ be a bijection $\lambda : P \to Q$.

A set $\mathcal{K}$ of k-tuples $(x_1, x_2, \ldots, x_k)$, $x_i \in P$, will be called a *polygonal presentation* over P compatible with λ if

(1) $(x_1, x_2, x_3, \ldots, x_k) \in \mathcal{K}$ implies that $(x_2, x_3, \ldots, x_k, x_1) \in \mathcal{K}$;
(2) given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \ldots, x_k) \in \mathcal{K}$ for some $x_3, \ldots, x_k$ if and only if x_2 and $\lambda(x_1)$ are incident in some G_i;
(3) given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \ldots, x_k) \in \mathcal{K}$ for at most one $x_3 \in P$.

If there exists such $\mathcal{K}$, we will call λ a *basic bijection*.

The polygonal presentations with $k = 3$, $n = 1$, and G_1 a generalized 3-gon have been listed in [4,5].

We can associate a polyhedron K on n vertices with each polygonal presentation $\mathcal{K}$ as follows: for every cyclic k-tuple $(x_1, x_2, x_3, \ldots, x_k)$ we take an oriented k-gon on the boundary of which the word $x_1 x_2 x_3 \cdots x_k$ is written. To obtain the polyhedron we identify the corresponding sides of our polygons, respecting orientation.

Lemma 2.2 [14]. *A polyhedron K which corresponds to a polygonal presentation $\mathcal{K}$ has graphs $G_1, G_2, \ldots, G_n$ as vertex-links.*

Now we construct two polygonal presentations with $k = 3$ and $n = 1$, but for which the graph G_1 is a generalized 4-gon. We denote the elements of P by x_i and the elements of Q by y_i, $i = 1, 2, \ldots, 15$. Let T_1 and T_2 be the two following sets of triples, and in both cases define the basic bijection $\lambda : P \to Q$ by $\lambda(x_i) = y_i$ for all $i = 1, 2, \ldots, 15$.

$$\begin{aligned} T_1\colon \big\{&(x_1,x_2,x_7), (x_1,x_8,x_{11}), (x_1,x_{14},x_5), (x_2,x_4,x_{13}), (x_{12},x_4,x_2),\\ &(x_4,x_9,x_3), (x_6,x_8,x_3), (x_{14},x_6,x_3), (x_{12},x_{10},x_5), (x_{13},x_{15},x_5),\\ &(x_{12},x_9,x_6), (x_{11},x_{10},x_7), (x_{14},x_{13},x_7), (x_9,x_{15},x_8), (x_{11},x_{15},x_{10})\big\}, \end{aligned}$$

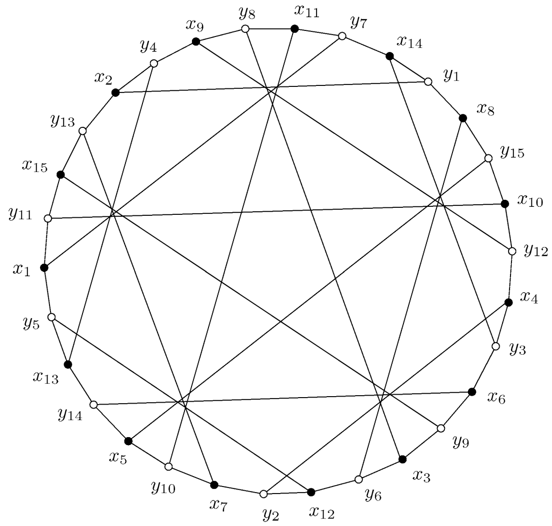

Fig. 1. Graph G_1 for T_1 with basic projection $\lambda(x_i) = y_i$.

$$T_2\colon \{(x_1, x_{10}, x_1), (x_1, x_{15}, x_2), (x_2, x_{11}, x_9), (x_2, x_{14}, x_3), (x_3, x_7, x_4), (x_3, x_{15}, x_{13}), (x_4, x_8, x_6), (x_4, x_{12}, x_{11}), (x_5, x_8, x_5), (x_5, x_{10}, x_{12}), (x_6, x_{14}, x_6), (x_7, x_{12}, x_7), (x_8, x_{13}, x_9), (x_9, x_{14}, x_{15}), (x_{10}, x_{13}, x_{11})\}.$$

We can draw the bipartite graph G_1 for T_1 (Fig. 1). For every triple (x_i, x_j, x_k) in T_1 the points y_i and x_j as well as y_j and x_k and also y_k and x_i have to be incident in the graph. For T_2 we obtain a similar graph, only with a different labeling of the points.

Let us check that these sets are desired polygonal presentations. Remark, that the smallest thick generalized 4-gon can be presented in the following way: its 'points' are pairs (i, j), where $i, j = 1, \dots, 6$, $i \neq j$ and 'lines' are triples (i_1, j_1), (i_2, j_2), (i_3, j_3) of those pairs, where i_1, i_2, i_3, j_1, j_2 and j_3 are all different. We mark pairs (i, j), where $i, j = 1, \dots, 6$, $i \neq j$ by x_1 to x_{15}. Now one can check by direct examination, that the graph G_1 is really the smallest thick generalized 4-gon. (See [13] for classification of generalized quadrangles.)

Definition 2.3. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two polygonal presentations with $k = 3$, $n = 1$, and for which the graph G_1 is a generalized 4-gon. Then $\mathcal{K}_1$ and $\mathcal{K}_2$ are *equivalent*, if there exists an automorphism of the generalized 4-gon which transforms the 4-gon of $\mathcal{K}_1$ to the 4-gon of $\mathcal{K}_2$.

In our case there is no such automorphism transforming T_1 to T_2, since in T_1 no element appears twice in one triple, but in T_2 there are triples of the form (x_i, x_j, x_i). Thus the polygonal presentations T_1 and T_2 are not equivalent.

For polygonal presentation T_i, $i = 1, 2$, take 15 oriented regular hyperbolic triangles with angles $\pi/4$, write words from the presentation on their boundaries and glue together sides with the same letters, respecting orientation. The result is a hyperbolic polyhedron with one vertex and 15 faces and its universal covering is a triangular hyperbolic building. The fundamental group Γ_i, $i = 1, 2$, of the polyhedron acts simply transitively on vertices of the building. The group Γ_i, $i = 1, 2$, has 15 generators and 15 relations, which come naturally from the polygonal presentation T_i, $i = 1, 2$.

For the first homology groups we get $H_1(\Gamma_1) = \mathbb{Z}/162\mathbb{Z}$ and $H_1(\Gamma_2) = \mathbb{Z}/9\mathbb{Z}$.

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