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Numerical Analysis/Calculus of Variations

A density result for the variation of a material with respect to small inclusions

Juan Casado-Díaz, Julio Couce-Calvo, José Domingo Martín-Gómez

Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, c/Tarfia s/n, 41012 Sevilla, Spain

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Abstract

We consider the family of materials obtained, via homogenization, by replacing a small portion, of size ε, of a fixed material by other materials. In a previous paper we have obtained a subset of the set of 'derivatives' of this family with respect to ε in $\varepsilon = 0$. In the present Note we prove that this set is, in fact, dense. This result can be applied, for example, to obtain optimality conditions for composite materials. ***To cite this article: J. Casado-Díaz et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Un résultat sur la variation d'un matériau en fonction de petites inclusions. On considère une famille de matériaux obtenus par homogénéisation consistant à remplacer une petite partie de matériau, de taille ε, par d'autres matériaux. Dans un article antérieur on a caractérisé un sous-ensemble de l'ensemble des « dérivées », par rapport à ε de cette famille, pour $\varepsilon = 0$. Dans cette Note on démontre que ce sous-ensemble est en fait dense. Le résultat peut être appliqué, par exemple, à l'obtention des conditions d'optimalité pour des matériaux composites. ***Pour citer cet article : J. Casado-Díaz et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Il est bien connu que les caractéristiques d'un matériau (par exemple en conductivité électrique ou thermique) sont normalement associées à une matrice A (matrice de diffusion). Quand on considère des matériaux composites, il est intéressant de déterminer les variations de cette matrice quand on change des petites parties du matériau par d'autres matériaux différents. Ce problème se pose, par exemple, dans l'étude des conditions d'optimalité pour A. Dans ce cas on considère habituellement des petites variations de la matrice A (voir [1–3,8–10]). Un autre exemple est la construction des directions de descente qui permettent d'améliorer A en introduisant d'autres matériaux. Puisque le mélange des matériaux est bien caractérisé par la théorie de l'homogénéisation (voir par exemple [1,3,5,7,8,10]) et puisque la H-convergence est un procédé local, la question à poser est la suivante : Pour $A, B_1, \ldots, B_n$ matrices définies positives, $\theta_1, \ldots, \theta_n \geqslant 0$, avec $\sum_{i=1}^n \theta_i = 1$, il s'agit d'obtenir l'ensemble $\mathcal{D}(A; \theta_1, \ldots, \theta_n; B_1, \ldots, B_n)$ de toutes les matrices D qui sont une limite de la forme

E-mail addresses: jcasadod@us.es (J. Casado-Díaz), couce@us.es (J. Couce-Calvo), jdmartin@us.es (J.D. Martín-Gómez).

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$$D = \lim_{\varepsilon \to 0} \frac{A^\varepsilon - A}{\varepsilon}, \tag{1}$$

où, pour chaque $\varepsilon \in (0,1)$, A^ε est obtenue par homogénéisation, en mélangeant les matériaux correspondants aux matrices $A, B_1, \dots, B_n$ avec des proportions respectives égales à $1-\varepsilon, \varepsilon\theta_1, \dots, \varepsilon\theta_n$.

Dans un article antérieur, [2], on a montré (pour $n = 1$, mais le cas général est analogue) qu'un sous-ensemble de $\mathcal{D}(A; \theta_1, \dots, \theta_n; B_1, \dots, B_n)$ est donné par l'ensemble $\mathcal{H}(A; \theta_1, \dots, \theta_n; B_1, \dots, B_n)$ de toutes les matrices H, telles qu'il existe $\omega_1, \dots, \omega_n \subset \mathbb{R}^N$, mesurables, disjoints, avec $|\omega_i| = \theta_i$, $i = 1, \dots, n$, et tel que

$$H\xi = \sum_{i=1}^{n} (B_i - A)\left(\theta_i \xi + \int_{\omega_i} \nabla w_\xi \, dz\right), \quad \forall \xi \in \mathbb{R}^N,$$

où w_ξ est la solution (unique à une constante additive près) de

$$-\operatorname{div}\left[A\chi_{\mathbb{R}^N \setminus \bigcup_{i=1}^n \omega_i} + \sum_{i=1}^{n} B_i \chi_{\omega_i}\right]\nabla w_\xi = \sum_{i=1}^{n} \operatorname{div}(B_i - A)\chi_{\omega_i}\xi \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \quad \nabla w_\xi \in L^2(\mathbb{R}^N)^N.$$

L'ensemble $\mathcal{H}(A; \theta_1, \dots, \theta_n; B_1, \dots, B_n)$ a été utilisé dans [2] pour obtenir des conditions d'optimalité sur A. De plus, on a aussi obtenu quelques éléments explicites de $\mathcal{H}(A; \theta_1, \dots, \theta_n; B_1, \dots, B_n)$.

L'objectif de cette Note est de montrer le résultat suivant :

Théorème 0.1. *L'ensemble $\mathcal{H}(A; \theta_1, \dots, \theta_n; B_1, \dots, B_n)$ est dense dans $\mathcal{D}(A; \theta_1, \dots, \theta_n; B_1, \dots, B_n)$.*

La démonstration utilise un résultat bien connu dû à Dal Maso et Kohn [1,4,5] établissant que les matrices obtenues par homogénéisation périodique sont denses dans l'ensemble obtenu par homogénéisation quelconque. Ceci donne une approximation du quotient qui apparaît dans (1), mais on remarque que cette approximation n'est pas dans $\mathcal{D}(A; \theta_1, \dots, \theta_n; B_1, \dots, B_n)$ et donc, par exemple, elle ne peut pas être utilisée comme direction admissible pour un problème d'optimalité.

1. Introduction and main result

It is well known that the characteristics of a material (for example, in heat or electric conductivity problems) are associated to a matrix A (the diffusion matrix). When we deal with composite materials, it is interesting to know the variation of this matrix when we replace a small portion of the material by other materials. This question arises, for example, in the study of optimality conditions for A, where to construct admissible directions, we consider small perturbations of A (see e.g. [1–3,8–10]). We also mention the construction of descent directions which permit to improve a material A by introducing other materials. Since the mixture of materials can be characterized via homogenization (see e.g. [1,3,5,7,8,10]) and this is a local process, the mathematical question is the following: Given $A, B_1, \dots, B_n$ definite positive matrices in $\mathbb{R}^N$, $\theta_1, \dots, \theta_n \geqslant 0$, with $\sum_{i=1}^n \theta_i = 1$, our problem is to obtain the set $\mathcal{D}(A; \theta_1, \dots, \theta_n; B_1, \dots, B_n)$ of all the possible matrices D which can be obtained as a limit of the form

$$D = \lim_{\varepsilon \to 0} \frac{A^\varepsilon - A}{\varepsilon}, \tag{2}$$

where, for every $\varepsilon \in (0,1)$, A^ε is obtained via homogenization, by mixing the materials corresponding to the matrices $A, B_1, \dots, B_n$ with respective proportions $1-\varepsilon, \varepsilon\theta_1, \dots, \varepsilon\theta_n$.

In a previous article, [2], we have proved (for $n = 1$, but the general case is analogous) that a subset of $\mathcal{D}(A; \theta_1, \dots, \theta_n; B_1, \dots, B_n)$ is given by the set $\mathcal{H}(A; \theta_1, \dots, \theta_n; B_1, \dots, B_n)$ of all the matrices H, such that there exist $\omega_1, \dots, \omega_n \subset \mathbb{R}^N$ measurable, disjoint, with $|\omega_i| = \theta_i$, $i = 1, \dots, n$, which satisfy:

$$H\xi = \sum_{i=1}^{n} (B_i - A)\left(\theta_i \xi + \int_{\omega_i} \nabla w_\xi \, dz\right), \quad \forall \xi \in \mathbb{R}^N,$$

where w_ξ is the solution (it is unique up to an additive constant) of

$$-\operatorname{div}\left[A\chi_{\mathbb{R}^N \setminus \bigcup_{i=1}^n \omega_i} + \sum_{i=1}^{n} B_i \chi_{\omega_i}\right]\nabla w_\xi = \sum_{i=1}^{n} \operatorname{div}(B_i - A)\chi_{\omega_i}\xi \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \quad \nabla w_\xi \in L^2(\mathbb{R}^N)^N. \tag{3}$$

The set $\mathcal{H}(A;\theta_1,\dots,\theta_n;B_1,\dots,B_n)$ has been used in [2] to obtain optimality conditions for A. Some explicit elements of $\mathcal{H}(A;\theta_1,\dots,\theta_n;B_1,\dots,B_n)$ have also been obtained in the mentioned paper.

We observe that, using a dilatation, $\mathcal{H}(A;\theta_1,\dots,\theta_n;B_1,\dots,B_n)$ can also be defined as the set of matrices H, such that there exist $\lambda>0$, and $\omega_1,\dots,\omega_n\subset\mathbb{R}^N$ measurable, disjoint, with $|\omega_i|=\lambda\theta_i$, $i=1,\dots,n$, which satisfy:

$$H\xi=\sum_{i=1}^n(B_i-A)\left(\theta_i\xi+\frac{1}{\lambda}\int_{\omega_i}\nabla w_\xi\,\mathrm{d}z\right),\tag{4}$$

where w_ξ is the solution of (3).

The goal of the present Note is to prove the following result:

Theorem 1.1. *The set $\mathcal{H}(A;\theta_1,\dots,\theta_n;B_1,\dots,B_n)$ is dense in $\mathcal{D}(A;\theta_1,\dots,\theta_n;B_1,\dots,B_n)$.*

The proof uses a well known result of Dal Maso and Kohn [1,4,5], which proves that the matrices obtained via periodic homogenization are dense in those which can be obtained by general homogenization. This provides an approximation of the quotient which appears in (2), but we remark that this approximation is not in $\mathcal{D}(A;\theta_1,\dots,\theta_n;B_1,\dots,B_n)$ and thus, for example, it cannot be used as an admissible direction for optimal problems.

2. Proof of Theorem 1.1

In the whole of the proof, we denote by C a generic constant, which can change from a line to another one. The proof is divided in three steps.

Step 1. Let D be in $\mathcal{D}(A;\theta_1,\dots,\theta_n;B_1,\dots,B_n)$, and consider a sequence A^ε of matrices obtained via homogenization, by mixing the materials corresponding to $A, B_1,\dots,B_n$ with respective proportions $1-\varepsilon, \varepsilon\theta_1,\dots,\varepsilon\theta_n$, and such that (2) is satisfied. Using the result of Dal Maso and Kohn mentioned above, we can assume that for every $\varepsilon>0$, there exist $\omega_i^\varepsilon\subset Y=(-\frac12,\frac12)^N$, measurable, with $|\omega_i^\varepsilon|=\varepsilon\theta_i$, $i=1,\dots,n$, such that defining $\widetilde{\omega}_i^\varepsilon=\bigcup_{k\in\mathbb{Z}^N}(\omega_i^\varepsilon+k)$ (the extension by Y-periodicity of $\omega_i^\varepsilon\subset Y$ to $\mathbb{R}^N$) $M^\varepsilon=A\chi_{\mathbb{R}^N\setminus\bigcup_{i=1}^n\omega_i^\varepsilon}+\sum_{i=1}^n B_i\chi_{\omega_i^\varepsilon}$, $\widetilde{M}^\varepsilon=A\chi_{\mathbb{R}^N\setminus\bigcup_{i=1}^n\widetilde{\omega}_i^\varepsilon}+\sum_{i=1}^n B_i\chi_{\widetilde{\omega}_i^\varepsilon}$ (remark that $M_\varepsilon=\widetilde{M}_\varepsilon$ in Y), we have

$$A^\varepsilon\xi=\int_Y M_\varepsilon\big(\nabla v_\xi^\varepsilon+\xi\big)\,\mathrm{d}y,\quad\forall\xi\in\mathbb{R}^N,\tag{5}$$

where v_ξ^ε is the solution of (it is unique up to a constant):

$$-\operatorname{div}\widetilde{M}_\varepsilon\nabla v_\xi^\varepsilon=\sum_{i=1}^n\operatorname{div}(B_i-A)\chi_{\widetilde{\omega}_i^\varepsilon}\xi\quad\text{in }\mathcal{D}'\big(\mathbb{R}^N\big),\quad v_\xi^\varepsilon\in H^1_\sharp(Y).\tag{6}$$

Here $H^1_\sharp(Y)$ denotes the space of functions of $H^1_{\mathrm{loc}}(\mathbb{R}^N)$, which are Y-periodic. A simple calculation shows

$$D^\varepsilon\xi=\frac{A^\varepsilon-A}{\varepsilon}\xi=\sum_{i=1}^n(B_i-A)\left(\theta_i\xi+\frac{1}{\varepsilon}\int_{\omega_i^\varepsilon}\nabla v_\xi^\varepsilon\,\mathrm{d}y\right).\tag{7}$$

Now, we define w_ξ^ε as the solution of

$$-\operatorname{div}M^\varepsilon\nabla w_\xi^\varepsilon=\sum_{i=1}^n\operatorname{div}(B_i-A)\chi_{\omega_i^\varepsilon}\xi\quad\text{in }\mathcal{D}'\big(\mathbb{R}^N\big),\quad\nabla w_\xi^\varepsilon\in L^2\big(\mathbb{R}^N\big)^N.\tag{8}$$

Then, from (4), the matrix H^ε, defined by:

$$H^\varepsilon\xi=\sum_{i=1}^n(B_i-A)\left(\theta_i\xi+\frac{1}{\varepsilon}\int_{\omega_i^\varepsilon}\nabla w_\xi^\varepsilon\,\mathrm{d}y\right),\tag{9}$$

belongs to $\mathcal{H}(A;\theta_1,\dots,\theta_n;B_1,\dots,B_n)$. In order to prove Theorem 1.1, it is enough to show that $(H^\varepsilon - D^\varepsilon)\xi$ tends to zero for every $\xi\in\mathbb{R}^N$, when ε tends to zero.

Step 2. For every $\rho>0$, we denote $Y_\rho=\{y\in Y\colon\ \mathrm{dist}(y,\partial Y)>\rho\}$. Let us assume that there exists $\delta>0$ such that $\bigcup_{i=1}^n\omega_i^\varepsilon\subset Y_\delta$. Taking into account that w_ξ^ε satisfies $-\operatorname{div}A\nabla w_\xi^\varepsilon=\operatorname{div}g^\varepsilon$ in $\mathcal{D}'(\mathbb{R}^N)$, with $g^\varepsilon=\sum_{i=1}^n(B_i-A)(\nabla w_\xi^\varepsilon+\xi)\chi_{\omega_i^\varepsilon}$, and that ∇w_ξ^ε belongs to $L^2(\mathbb{R}^N)^N$, we deduce that (a representative of w_ξ^ε is) $w_\xi^\varepsilon=\nabla K_A*g^\varepsilon$, with K_A the fundamental solution of $-\operatorname{div}A\nabla$. Since w_ξ^ε satisfies (take w_ξ^ε as test function in (8))

$$\|\nabla w_\xi^\varepsilon\|_{L^2(\mathbb{R}^N)^N}\leqslant C\sqrt{\varepsilon},\qquad \nabla K_A(z)=\mu\frac{L^tLz}{|Lz|^N},$$

with $\mu\in\mathbb{R}$, $L\in\mathcal{M}_N$ non-singular, and the support of g^ε is contained in $\bigcup_{i=1}^n\omega_i^\varepsilon$, we deduce:

$$\left|w_\xi^\varepsilon(z)\right|\leqslant C\frac{\varepsilon}{\delta^{N-1}},\quad \left|\nabla w_\xi^\varepsilon(z)\right|\leqslant C\frac{\varepsilon}{\delta^N},\quad \forall z \text{ with } \mathrm{dist}\left(z,\bigcup_{i=1}^n\omega_i^\varepsilon\right)>\frac{\delta}{2}. \tag{10}$$

We take $\psi\in C_0^\infty(\mathbb{R}^N)$ a cut-off function such that

$$\psi=1\quad\text{in } Y_{\delta/2},\quad \mathrm{supp}(\psi)\subset\overline{Y},\quad |\nabla\psi|\leqslant\frac{C}{\delta}\quad\text{in } \mathbb{R}^N. \tag{11}$$

From (10) and (11), we deduce:

$$\left\|\nabla\left(w_\xi^\varepsilon\psi-w_\xi^\varepsilon\right)\right\|_{L^2(Y)^N}\leqslant C\frac{\varepsilon}{\delta^{N-1/2}}. \tag{12}$$

On the other hand, for every $\varphi\in H^1(Y)$, we have:

$$\int_Y M^\varepsilon\nabla\left(\psi w_\xi^\varepsilon\right)\nabla\varphi\,\mathrm{d}y=\int_Y M^\varepsilon\nabla w_\xi^\varepsilon\nabla\left(\psi\left[\varphi-\int_Y\varphi\,\mathrm{d}z\right]\right)\mathrm{d}y-\int_Y M^\varepsilon\nabla w_\xi^\varepsilon\nabla\psi\left[\varphi-\int_Y\varphi\,\mathrm{d}z\right]\mathrm{d}y$$
$$+\int_Y M^\varepsilon\nabla\psi\nabla\varphi\,w_\xi^\varepsilon\,\mathrm{d}y.$$

Using (8), and $\psi=1$ in $\bigcup_{i=1}^n\omega_i^\varepsilon$, the first term on the right-hand side is

$$\sum_{i=1}^n(A-B_i)\xi\int_{\omega_i^\varepsilon}\nabla\varphi\,\mathrm{d}y.$$

The second and third terms on the right-hand side can be estimated by using (10), (11) and Poincaré–Wirtinger's inequality. We then get:

$$\left|\int_Y M^\varepsilon\nabla\left(\psi w_\xi^\varepsilon\right)\nabla\varphi\,\mathrm{d}y-\sum_{i=1}^n(A-B_i)\xi\int_{\omega_i^\varepsilon}\nabla\varphi\,\mathrm{d}y\right|\leqslant C\frac{\varepsilon}{\delta^{N+1/2}}\|\nabla\varphi\|_{L^2(Y)^N},\quad\forall\varphi\in H^1(Y),$$

and then, from (6), we get:

$$\left|\int_Y M^\varepsilon\nabla\left(\psi w_\xi^\varepsilon-v_\xi^\varepsilon\right)\nabla\varphi\,\mathrm{d}y\right|\leqslant C\frac{\varepsilon}{\delta^{N+1/2}}\|\nabla\varphi\|_{L^2(Y)^N},\quad\forall\varphi\in H^1_\sharp(Y).$$

Taking here $\varphi=\psi w_\xi^\varepsilon-v_\xi^\varepsilon$, we deduce $\|\nabla(\psi w_\xi^\varepsilon-v_\xi^\varepsilon)\|_{L^2(Y)^N}\leqslant C\varepsilon\delta^{-N-1/2}$, which joining to (12) shows

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_Y\left|\nabla\left(w_\xi^\varepsilon-v_\xi^\varepsilon\right)\right|^2\mathrm{d}y=0,\quad\forall\xi\in\mathbb{R}^N. \tag{13}$$

From (7) and (9) we then conclude that $(H^\varepsilon-D^\varepsilon)\xi$ tends to zero, for every $\xi\in\mathbb{R}^N$.

Step 3. Let us now consider the general case.

We fix $\delta \in (0,1)$. Since in (5) and (6), ω_i^ε can be replaced by $\omega_i^\varepsilon + a$, for every $a \in \mathbb{R}^N$, then, choosing a appropriately, we can assume that there exists $C > 0$ which does not depend on ε nor on δ, such that

$$\left| \bigcup_{i=1}^{n} \big(\omega_i^\varepsilon \cap (Y \setminus Y_\delta)\big) \right| \leqslant C\delta\varepsilon. \tag{14}$$

We take

$$\omega_i^{\varepsilon,\delta} = \omega_i^\varepsilon \cap Y_\delta, \qquad M^{\varepsilon,\delta} = A\chi_{\mathbb{R}^N \setminus \bigcup_{i=1}^n \omega_i^{\varepsilon,\delta}} + \sum_{i=1}^{n} B_i \chi_{\omega_i^{\varepsilon,\delta}}.$$

As above, we also denote by $\widetilde{\omega}_i^{\varepsilon,\delta}$, the extension by Y-periodicity of $\omega_i^{\varepsilon,\delta} \subset Y$ to $\mathbb{R}^N$ and by $\widetilde{M}^{\varepsilon,\delta}$, the extension by Y-periodicity of $M^{\varepsilon,\delta}\chi_Y$ to $\mathbb{R}^N$. We define $w_\xi^{\varepsilon,\delta}$, $v_\xi^{\varepsilon,\delta}$ as the solutions of

$$-\operatorname{div} M^{\varepsilon,\delta} \nabla w_\xi^{\varepsilon,\delta} = \sum_{i=1}^{n} \operatorname{div}(B_i - A)\chi_{\omega_i^{\varepsilon,\delta}}\xi \quad \text{in } \mathcal{D}'\big(\mathbb{R}^N\big), \quad \nabla w_\xi^{\varepsilon,\delta} \in L^2\big(\mathbb{R}^N\big)^N, \tag{15}$$

$$-\operatorname{div} \widetilde{M}^{\varepsilon,\delta} \nabla v_\xi^{\varepsilon,\delta} = \sum_{i=1}^{n} \operatorname{div}(B_i - A)\chi_{\widetilde{\omega}_i^{\varepsilon,\delta}}\xi \quad \text{in } \mathcal{D}'\big(\mathbb{R}^N\big), \quad v_\xi^{\varepsilon,\delta} \in H^1_\sharp(Y). \tag{16}$$

From Step 2 (see (13)), we know

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_Y \big|\nabla\big(w_\xi^{\varepsilon,\delta} - v_\xi^{\varepsilon,\delta}\big)\big|^2 \,\mathrm{d}y = 0, \quad \forall \xi \in \mathbb{R}^N. \tag{17}$$

Let us now estimate $w_\xi^\varepsilon - w_\xi^{\varepsilon,\delta}$. Taking it as test function in the difference of (8) and (15) we get:

$$\int_{\mathbb{R}^N} M^\varepsilon \nabla\big(w_\xi^\varepsilon - w_\xi^{\varepsilon,\delta}\big)\nabla\big(w_\xi^\varepsilon - w_\xi^{\varepsilon,\delta}\big)\,\mathrm{d}x = \sum_{i=1}^{n} \int_{\omega_i^\varepsilon \setminus \omega_i^{\varepsilon,\delta}} (A - B_i)\nabla\big(w_\xi^\varepsilon + \xi\big)\nabla\big(w_\xi^\varepsilon - w_\xi^{\varepsilon,\delta}\big)\,\mathrm{d}x,$$

which gives

$$\int_{\mathbb{R}^N} \big|\nabla\big(w_\xi^\varepsilon - w_\xi^{\varepsilon,\delta}\big)\big|^2 \,\mathrm{d}x \leqslant C \int_{\bigcup_{i=1}^n (\omega_i^\varepsilon \cap (Y \setminus Y_\delta))} \big(1 + \big|\nabla w_\xi^\varepsilon\big|\big)^2 \,\mathrm{d}x. \tag{18}$$

From Meyers' theorem [6] applied to (8), we know that there exists $p > 2$ such that

$$\|\nabla w_\xi^\varepsilon\|_{L^p(Y)^N} \leqslant C\left(\left\|\sum_{i=1}^{n} \operatorname{div}(B_i - A)\chi_{\omega_i^{\varepsilon,\delta}}\xi\right\|_{W^{-1,p}(2Y)} + \|\nabla w_\xi^\varepsilon\|_{L^2(2Y)^N}\right),$$

and as w_ξ^ε satisfies $\|\nabla w_\xi^\varepsilon\|_{L^2(\mathbb{R}^N)^N} \leqslant C\sqrt{\varepsilon}$, we deduce $\|\nabla w_\xi^\varepsilon\|_{L^p(Y)^N} \leqslant C\sqrt[p]{\varepsilon}$. Thus, estimate (18) shows

$$\big\|\nabla\big(w_\xi^\varepsilon - w_\xi^{\varepsilon,\delta}\big)\big\|_{L^2(\mathbb{R}^N)^N} \leqslant C\delta^{(p-2)/(2p)}\varepsilon^{1/2}.$$

A similar reasoning proves

$$\big\|\nabla\big(v_\xi^\varepsilon - v_\xi^{\varepsilon,\delta}\big)\big\|_{L^2(Y)^N} \leqslant C\delta^{(p-2)/(2p)}\varepsilon^{1/2}.$$

These estimates and (17), then prove

$$\limsup_{\varepsilon \to 0} \big\|\nabla\big(v_\xi^\varepsilon - v_\xi^{\varepsilon,\delta}\big)\big\|_{L^2(Y)^N} \leqslant C\delta^{(p-2)/(2p)}\varepsilon^{1/2}.$$

So, from (7) and (9), we deduce that

$$\limsup_{\varepsilon \to 0} \big|\big(H^\varepsilon - D^\varepsilon\big)\xi\big| \leqslant C\delta^{(p-2)/(2p)}, \quad \forall \delta \in (0,1)$$

and then $(H^\varepsilon - D^\varepsilon)\xi$ tends to zero, for every $\xi \in \mathbb{R}^N$.

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