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Numerical Analysis

Arbitrary high-order schemes for the linear advection and wave equations: application to hydrodynamics and aeroacoustics

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Abstract

We present here an extension to any order of accuracy of the schemes proposed in Daru and Tenaud [J. Comput. Phys. 193 (2) (2004) 563–594] for the linear advection equation in 1D. Such schemes are then used for a high-order generalization of the Godunov method in the case of the wave equation and the locally linearized Euler equations. ***To cite this article: S. Del Pino, H. Jourdren, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Schémas d'ordre arbitraire pour l'advection linéaire et l'équation des ondes : application à l'hydrodynamique et l'aéro-acoustique. On propose ici une extension à un ordre arbitraire des schémas proposés dans Daru et Tenaud [J. Comput. Phys. 193 (2) (2004) 563–594] pour l'advection à vitesse uniforme en 1D. Les schémas obtenus sont alors utilisés pour une généralisation d'ordre élevée du schéma de Godunov dans le cas de l'équation des ondes et des équations d'Euler linéarisées localement. ***Pour citer cet article : S. Del Pino, H. Jourdren, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

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L'étude de la propagation d'ondes acoustiques atmosphériques est un problème difficile. Les variations de pression génèrent des variations de la vitesse du son, conduisant à un raidissement des ondes qui dégénèrent en chocs sur de longues distances. Ce phénomène peut justifier la résolution des équations d'Euler pour une bonne modélisation du problème. Cette Note propose un schéma numérique adéquat, car de haut degré pour les régimes acoustiques ou de transport pur. Dans la première partie, on étend les schémas proposés dans [1] pour l'équation d'advection à des ordres arbitraires en espace et en temps. On décrit ensuite l'extension naturelle de ces schémas au cas des systèmes hyperboliques linéaires à coefficients constants, en détaillant le cas de l'équation des ondes. Enfin, on donne une adaptation de ces schémas au cas des équations d'Euler, tant en coordonnées de Lagrange qu'en coordonnées d'Euler.

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Dans [1], Daru et Tenaud ont établi des schémas jusqu'à l'ordre 7 en espace et en temps pour résoudre (1). Comme pour le schéma de Lax–Wendroff [6], ces schémas dits à un pas s'appuient sur l'utilisation de la procédure de Cauchy–Kowalewski (2) pour monter en ordre en temps. La Proposition 2.1 donne les flux de schémas numériques à des ordres arbitraires N en espace et en temps. Les résultats obtenus à l'aide de ces schémas sont spectaculaires lorsque la condition initiale u_0 est une fonction régulière. Pour des solutions non régulières on recourt aux limiteurs (voir [1]).

Il est très simple de construire des schémas de haut degré en espace et en temps pour les systèmes linéaires de lois de conservation à coefficients constants, puisqu'ils vérifient (7). Ils peuvent donc être approchés à l'ordre N en espace et en temps en utilisant (5) et (6), l'ordre d'approximation étant préservé par linéarité. Une application intéressante est l'équation des ondes à coefficients constants en 1D (8).

On peut construire un schéma d'ordre élevé pour les régimes asymptotiques onde et transport des équations d'Euler. Pour cela, on choisit une approche Lagrange-Advection. La raison essentielle est que, dans le cas isentropique (10), les équations d'Euler en coordonnées de Lagrange sont proches de l'équation des ondes (8) que nous savons approcher à l'ordre N. L'écriture du schéma, faite dans le cas polytropique, repose sur la connaissance des invariants de Riemann. Le schéma proposé est construit comme une généralisation du schéma de Godunov en solveur de Riemann double détente, autour d'états linéarisés. Cette approche permet de déduire facilement les flux numériques en variables primitives des flux en variables caractéristiques. L'advection s'appuie sur une linéarisation locale de (15) figeant u aux faces de la maille.

On illustre le bon comportement de ce schéma à l'aide d'expériences numériques choisies. On simule d'abord un tube à choc de Sod afin de valider le schéma pour les régimes non linéaires des équations d'Euler. On s'intéresse ensuite au cas de la dégénérescence des équations vers le transport pur, la grandeur transportée, à vitesse uniforme, étant la densité. Il est aisé de vérifier que les résultats obtenus sont rigoureusement identiques à ceux obtenus par la résolution de l'équation de transport. On montre enfin l'intérêt de cette technique pour la résolution d'un problème acoustique. À l'aide du schéma d'ordre 10, 7 points par longueur d'onde suffisent pour maintenir l'amplitude du signal à son niveau de départ, sur un grand nombre de longueurs d'onde.

1. Introduction

Study of acoustic wave propagation in the atmosphere is a challenging problem. Pressure variations generate variations of the sound speed, leading to shocks over long distances. Such phenomenon may justify the use of the Euler equations, instead of the simpler wave equation. This Note proposes a numerical scheme well-suited to this problem, since relying on a very high-order scheme for both acoustic and transport regimes of the Euler equations. In a first part, we shall extend the schemes proposed in [1] to any order of accuracy in the case of the advection equation. In a second part, we shall discuss its natural extension to linear hyperbolic systems with constant coefficients, with special emphasis on the wave equation. We shall then consider the 1D Euler equations in both Lagrangian and Eulerian coordinates.

2. Constant linear advection equation

In [1], Daru and Tenaud have established up to the seventh order in time and space one-step schemes to discretize the 1D linear advection equation with constant coefficient

$$\partial_t u + a\partial_x u = 0, \quad \text{in } \mathbb{R}, \quad \text{with } u(\cdot,0) = u_0, \text{ and } a \in \mathbb{R}^+. \tag{1}$$

The proposed schemes are called *one-step* since the high-order approximation in time is not obtained using a Runge–Kutta method, limited to the fourth order with nonlinear flux corrections (Osher–Shu), but thanks to the Cauchy–Kowalewski procedure applied to (1) that reads

$$\partial_t^k u = (-a)^k \partial_x^k u, \quad 0 \leqslant k \leqslant n, \text{ if } u_0 \in C^n(\mathbb{R}). \tag{2}$$

The derivation, previously used for the Lax–Wendroff scheme [6], consists in computing the equivalent equation of an order $N-1$ scheme, replacing the time derivatives of order $N-1$ error using (2), and then discretizing them at order 1 to get an approximation of the equation at order N.

Denoting $x_j = j\Delta x$, $t_n = n\Delta t$ and $u_j^n = u(x_j, t_n)$ Taylor expansions of u at point (x_j, t_n) lead to

$$\partial_x u = \frac{u_j^n - u_{j-1}^n}{\Delta x} - \sum_{k=2}^{N} \frac{(-\Delta x)^{k-1}}{k!} \partial_x^k u + \mathrm{O}(\Delta x^N), \quad \text{and}$$

$$\partial_t u = \frac{u_j^{n+1} - u_j^n}{\Delta t} - \sum_{k=2}^{N} \frac{\Delta t^{k-1}}{k!} \partial_t^k u + \mathrm{O}(\Delta t^N) = \frac{u_j^{n+1} - u_j^n}{\Delta t} + a \sum_{k=2}^{N} \frac{(-\nu \Delta x)^{k-1}}{k!} \partial_x^k u + \mathrm{O}(\Delta t^N), \tag{3}$$

where $\nu = a\Delta t/\Delta x$ and a is supposed positive to ensure an upwind discretization. Then

$$\partial_t u + a\partial_x u = \frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} - a\sum_{k=2}^{N} \frac{(1-\nu^{k-1})(-\Delta x)^{k-1}}{k!} \partial_x^k u + \mathrm{O}(\Delta x^N) + \mathrm{O}(\Delta t^N). \tag{4}$$

Building an order N scheme only requires to discretize $\partial_x^k u$ at order $N-k+1$ for $k \in \{2, \dots, N\}$.

Proposition 2.1. *The scheme* $u_j^{n+1} = u_j^n - a\frac{\Delta t}{\Delta x}(F_{j+1/2}^N - F_{j-1/2}^N)$, *defined by* $F_{j+1/2}^1 = u_j^n$ *and*

$$F_{j+1/2}^N = F_{j+1/2}^{N-1} - \frac{1}{N!}\left(\prod_{i=-m,\, i\neq 0}^{M} (\nu + i)\right)\left(\sum_{k=0}^{N-1} (-1)^{k+N} C_{N-1}^k u_{j+m-k}^n\right), \tag{5}$$

with $m = \left[\frac{N}{2}\right]$, $M = \left[\frac{N-1}{2}\right]$ *and* $C_n^p = \frac{n!}{p!(n-p)!}$, *is order N accurate, in space and time, approximating* (1).

Those schemes are very efficient for smooth solutions of (1). For instance, if $u_0(x) = \sin^4(\pi x)$ a 300 vertices mesh of $]-1, 1[$ is enough to approximate the solution at the *machine-precision* using 64 bits floating numbers, with the eleventh-order scheme. When the solution is not smooth, various flux limiting strategies can be successfully applied. For instance, from [1] building a TVD flux only requires to enforce $F_{j+1/2}^{\mathrm{TVD}} \in [\![u_j^n, u_{j+1}^n]\!] \cap [\![u_j^n, u_j^{\mathrm{UL}}]\!]$, where $u_j^{\mathrm{UL}} = u_j^n + \frac{1-\nu}{\nu}(u_j^n - u_{j-1}^n)$ and $[\![\alpha, \beta]\!] = [\min(\alpha, \beta), \max(\alpha, \beta)]$.

We complete this part by giving the recursive formula in the case $a < 0$: $F_{j+1/2}^1 = u_{j+1}^n$ and

$$F_{j+1/2}^N = F_{j+1/2}^{N-1} - \frac{1}{N!}\left(\prod_{i=-M,\ i\neq 0}^{m} (\nu + i)\right)\left(\sum_{k=0}^{N-1} (-1)^{k+N} C_{N-1}^k u_{j+M-k+1}^n\right), \tag{6}$$

with the TVD condition $F_{j+1/2}^{\mathrm{TVD}} \in [\![u_j^n, u_{j+1}^n]\!] \cap [\![u_{j+1}^n, u_{j+1}^{\mathrm{UL}}]\!]$, where $u_{j+1}^{\mathrm{UL}} = u_{j+1}^n - \frac{1+\nu}{\nu}(u_{j+1}^n - u_{j+2}^n)$.

3. Linear hyperbolic systems of conservation laws with constant coefficients in 1D

In 1D, linear strictly hyperbolic systems of conservation laws with constant coefficients can be rewritten as systems of decoupled 1D transport equations. Let $\mathbf{A} \in \mathbb{R}^{p\times p}$, with $\mathbf{A}\mathbf{r}_k = \lambda_k \mathbf{r}_k$, $\forall k \in \{1, \dots, p\}$, and $\lambda_1 < \cdots < \lambda_p$. Introducing (w_k) such that $\mathbf{u} = \sum_{k=1}^{p} w_k \mathbf{r}_k$, we have

$$\partial_t \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} = 0 \quad \Leftrightarrow \quad \partial_t w_k + \lambda_k \partial_t w_k = 0, \quad \forall k \in \{1, \dots, p\}. \tag{7}$$

Using (5) and (6), it is possible to approximate each w_k at a prescribed order N in both time and space. The linearity of the system permits to reconstruct an approximation of $\mathbf{u}$ at the same order of accuracy.

This procedure applies easily to the interesting case of the 1D acoustic wave equation:

$$\begin{cases} \partial_t u + \frac{1}{\rho_0}\partial_x p = 0, \\ \partial_t p + \rho_0 c_0^2 \partial_x u = 0, \end{cases} \quad \text{where } \rho_0, c_0 \in \mathbb{R}_+^* \text{ are the density and the sound velocity.} \tag{8}$$

The Riemann invariants associated to (u, p) are $w_\pm = u \pm \frac{1}{\rho_0 c_0} p$. They satisfy two decoupled advection equations $\partial_t w_\pm \pm c_0 \partial_x w_\pm = 0$, that can be solved at order N in both space and time using (5) and (6). High-order approximations of u and p are reconstructed using $u = \frac{1}{2}(w_+ + w_-)$ and $p = \frac{\rho_0 c_0}{2}(w_+ - w_-)$.

4. 1D Euler equations

The aim of this part is to introduce a class of numerical schemes for the 1D Euler equations that takes advantage of the previous high-order advection schemes.

4.1. Lagrangian scheme

We first consider the 1D Euler equations written in Lagrangian coordinates

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m p = 0, \\ \partial_t e + \partial_m (pu) = 0, \end{cases} \quad \text{where } \tau = \tfrac{1}{\rho} \text{ and } \mathrm{d}m = \rho_0\,\mathrm{d}x \text{ defines the mass variable } m, \ (\rho_0 = \rho(\cdot, 0)), \tag{9}$$

u is the velocity, e the total energy and $p = p(\tau, e - \frac{u^2}{2})$ the pressure. See for instance [3] for details.

For smooth isentropic flows, the energy equation may be omitted since $p = p(\tau)$ leading to

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m p = 0, \end{cases} \quad \text{or equivalently,} \quad \begin{cases} \partial_t u + \partial_m p = 0, \\ \partial_t p + (\rho c)^2 \partial_m u = 0. \end{cases} \tag{10}$$

This is close to (8), especially when ρ and $c = \sqrt{\frac{\partial p}{\partial \rho}|_s}$ have small variations. Instead of the two variables $w_\pm$ introduced for (8), the Riemann invariants $J_\pm = u \pm \int \frac{\mathrm{d}p}{\rho c}$ that write $J_\pm = u \pm \frac{2}{\gamma - 1} c$ for polytropic gases (see [5]) may be used. $J_\pm$ satisfy two advection equations easily derived from (10)

$$\partial_t J_\pm \pm \rho c \partial_m J_\pm = 0. \tag{11}$$

At this step, we must precise that we will not derive here high-order schemes on the full Euler equations but on local linearizations of the equations. During any time step, c and ρ will be treated as constants. In this context, if ρ_0 is updated after each time step, we will replace (11) by $\partial_t J_\pm \pm c \partial_x J_\pm = 0$.

For smooth (u, p), these equations may be approximated using the schemes defined by (5), (6) and

$$J_{\pm}{}_j^{n+1} = J_{\pm}{}_j^n \mp c_j^n \frac{\Delta t}{\Delta x} \big(J_{\pm}{}_{j+1/2}^n - J_{\pm}{}_{j-1/2}^n \big). \tag{12}$$

For discontinuous solutions of (9), a *conservative* scheme is built by introducing velocity and pressure variables $(u_{j+1/2}, p_{j+1/2})$ related to the Riemann invariants $J_{\pm}{}_{j+1/2}^n$ for each numerical flux stencil[1] by

$$J_{+}{}_{j+1/2}^n = u_{j+1/2} + \frac{2}{\gamma - 1} \sqrt{\frac{\gamma {p_j^n}^{1/\gamma}}{\rho_j^n} {p_{j+1/2}}^{\frac{\gamma-1}{2\gamma}}}, \quad J_{-}{}_{j+1/2}^n = u_{j+1/2} - \frac{2}{\gamma - 1} \sqrt{\frac{\gamma {p_{j+1}^n}^{1/\gamma}}{\rho_{j+1}^n} {p_{j+1/2}}^{\frac{\gamma-1}{2\gamma}}}. \tag{13}$$

Using (13), $u_{j+1/2}$ and $p_{j+1/2}$ are easily obtained for use in the Lagrangian discretization:

$$\begin{cases} \tau_j^{n+1} = \tau_j^n + \frac{\Delta t}{\rho_j \Delta x} (u_{j+1/2} - u_{j-1/2}), \\ u_j^{n+1} = u_j^n - \frac{\Delta t}{\rho_j \Delta x} (p_{j+1/2} - p_{j-1/2}), \\ e_j^{n+1} = e_j^n - \frac{\Delta t}{\rho_j \Delta x} (p_{j+1/2} u_{j+1/2} - p_{j-1/2} u_{j-1/2}). \end{cases} \tag{14}$$

The scheme defined by (5)-(6)-(12)-(13)-(14) is a high-order generalization of the Godunov scheme [4] on the linearized state, upwind values for $J_{\pm}{}_{j+1/2}^n$ corresponding to the double-rarefaction Riemann solver.

[1] Upwind thermodynamic values in formulae (13) also apply to cell-centered $J_{\pm}{}_k^n$ of the $j + 1/2$ numerical flux stencil ($w_+ = u + (\rho_j c_j)^{-1} p$ and $w_- = u - (\rho_{j+1} c_{j+1})^{-1} p$ for the simpler wave equation Riemann invariants).

4.2. *Eulerian scheme*

In Eulerian coordinates, the advection step consists in solving

$$\partial_t \rho + u\partial_x \rho = 0, \quad \partial_t(\rho u) + u\partial_x(\rho u) = 0, \quad \text{and} \quad \partial_t(\rho e) + u\partial_x(\rho e) = 0. \tag{15}$$

Being a set of transport equations, it can be solved using (5) and (6) by considering that the advection velocity u is constant on each numerical flux stencil. The numerical choice of u is $u_{j+1/2}$, the flux computed during the Lagrangian step. This leads finally to the following Eulerian scheme:

$$\begin{cases} \rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{\Delta x}\big(\overline{\rho_{j+1/2}}\, u_{j+1/2} - \overline{\rho_{j-1/2}}\, u_{j-1/2}\big), \\ (\rho u)_j^{n+1} = (\rho u)_j^n - \frac{\Delta t}{\Delta x}\big(\overline{(\rho u)_{j+1/2}}\, u_{j+1/2} - \overline{(\rho u)_{j-1/2}}\, u_{j-1/2} + p_{j+1/2} - p_{j-1/2}\big), \\ (\rho e)_j^{n+1} = (\rho e)_j^n - \frac{\Delta t}{\Delta x}\big(\overline{(\rho e)_{j+1/2}}\, u_{j+1/2} - \overline{(\rho e)_{j-1/2}}\, u_{j-1/2} + u_{j+1/2} p_{j+1/2} - u_{j-1/2} p_{j-1/2}\big), \end{cases} \tag{16}$$

where the over-lined terms are related to advection fluxes and the others to the Lagrangian fluxes, providing a high-order complementary approach to [2].

5. Numerical experiments

To enlighten the efficiency of the proposed Eulerian scheme, we chose here three relevant test cases.

The first case is the classical Sod shock tube. Fig. 1(a) shows a good behavior of the scheme to treat the shock, the contact discontinuity and the rarefaction wave, even if the solution is not smooth at those points. The tenth-order scheme is used with the TVD limiter.

The second case is the advection of the nonregular initial density profile

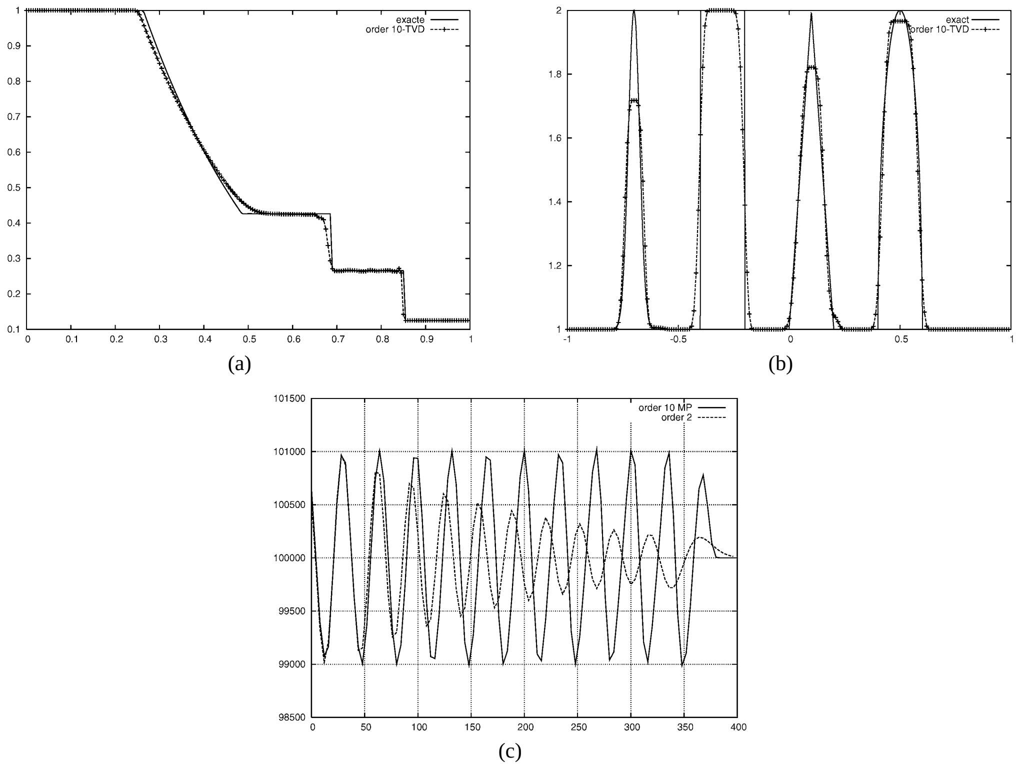

Fig. 1. Numerical experiments on the Euler equations with the proposed schemes with CFL = 0.6. (a) Sod shock tube, 200 vertices, tenth-order scheme with TVD limiter. Density at time $t = 0.2$ s. (b) Advection limit, 200 vertices, tenth-order TVD, numerical and exact solution after 100 revolutions. (c) Acoustic limit, 100 vertices, tenth and second-order MP: 7 vertices per wavelength (pressure).

Fig. 1. Résultats numériques pour les équations d'Euler avec les schémas proposés à CFL = 0, 6.

$$\rho_0(x) = 1 + \chi_{(-0.8,-0.6)}\,\mathrm{e}^{-\log(2)\frac{(x+0,7)^2}{0,0009}} + \chi_{(-0.4,-0.2)} + \chi_{(0,0.2)}\bigl(1 - \bigl|10(x-0,1)\bigr|\bigr) + \chi_{(0.4,0.6)}\bigl(1 - 100(x-0,5)^2\bigr)^{1/2}. \tag{17}$$

The test initializes the advection speed to 1 and uses periodic boundary conditions. The Euler equations are used to solve the problem. The scheme computes *exactly* the same solution as if the linear advection equation was considered. Fig. 1(b) shows the dissipation after 100 revolutions (tenth-order TVD).

We finally look at the acoustic limit of the 1D Euler equations. A 10 Hz signal of 10^3 Pa is imposed on the left boundary ($L = 400$ m, $p = 10^5$ Pa, $\rho = 1230$ kg/m^3, $\gamma = 1.4$). The signal must propagate without any lose of amplitude. Here, the tenth-order MP limited (see [7] and [1]) scheme preserves the amplitude with only 7 vertices per wavelength, while the second-order MP limited scheme shows a lot of dissipation.

6. Conclusion

In this Note we first extended to any order of accuracy the schemes of Daru and Tenaud for the linear advection equation, with an application to the wave equation. As emphasized in [1], such schemes are stable for Courant number $0 < \nu \leqslant 1$, attested by numerous numerical experiments (not proved yet).

In a second part, we proposed a generalization of the approach to the Euler equations. The main advantage of the resulting Godunov-type hydrodynamic schemes is that, in acoustic and transport regimes, they are formally of order N, while behaving pretty well in nonlinear regimes.

The proposed schemes have been tested successfully by P. Have[2] in 2D, using direction splitting, to solve efficiently acoustic problems with variable coefficients.

Ongoing work articulates around two main axis: first, in applying the Cauchy–Kowalewski procedure to build high-order finite differences schemes in the case of nonlinear equations, both scalar and systems. The second direction is a multi-dimensional extension without directional splitting.

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