Available online at www.sciencedirect.com

SCIENCE DIRECT®

C. R. Acad. Sci. Paris, Ser. I 342 (2006) 501–503

COMPTES RENDUS MATHEMATIQUE

http://france.elsevier.com/direct/CRASS1/

Functional Analysis

Maharam's problem

Michel Talagrand

Équipe d'analyse de l'institut mathématique, université Paris VI, boîte 186, 4, place Jussieu, 75230 Paris cedex 05, France

Received 27 January 2006; accepted 30 January 2006

Available online 3 March 2006

Presented by Gilles Pisier

Abstract

We solve Maharam's problem [D. Maharam, An algebraic characterization of measure algebras, Ann. Math. 48 (1947) 154–167. [3]], also known as the Control Measure Problem. We construct a non-zero exhaustive submeasure on the algebra of clopen sets of the Cantor set that is not absolutely continuous with respect to a measure. ***To cite this article: M. Talagrand, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Le problème de Maharam. Nous construisons une sous-mesure exhaustive non nulle sur l'algèbre des ouverts-fermés de l'ensemble de Cantor qui n'est absolument continue par rapport à aucune mesure. Ceci résout un problème posé par D. Maharam [D. Maharam, An algebraic characterization of measure algebras, Ann. Math. 48 (1947) 154–167. [3]], en 1947, aussi connu sous le nom de problème des mesures de contrôle. ***Pour citer cet article : M. Talagrand, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

Given a Boolean algebra $\mathcal{B}$ of sets, a map $\nu : \mathcal{B} \to \mathbb{R}^+$ is called a *submeasure* if it satisfies the following properties:

(1.1) $\nu(\varnothing) = 0$,
(1.2) $A \subset B,\ A, B \in \mathcal{B} \Rightarrow \nu(A) \leqslant \nu(B)$,
(1.3) $A, B \in \mathcal{B} \Rightarrow \nu(A \cup B) \leqslant \nu(A) + \nu(B)$.

If moreover $\nu(A \cup B) = \nu(A) + \nu(B)$ whenever A and B are disjoint ν is called a *measure*. We say that ν is *exhaustive* if $\lim_{n\to\infty} \nu(E_n) = 0$ for each disjoint sequence (E_n) of $\mathcal{B}$ (that is, $E_n \cap E_m = \varnothing$ if $n \neq m$). It is obvious that a measure is exhaustive. Given two submeasures ν_1 and ν_2, we say that ν_1 is absolutely continuous with respect to ν_2 if

$$\forall \varepsilon > 0,\ \exists \alpha > 0,\ \nu_2(B) < \alpha \Rightarrow \nu_1(B) \leqslant \varepsilon.$$

In other words, for a certain function $f : \mathbb{R}^+ \to \mathbb{R}^+$ continuous at zero, we have $\nu_1(B) \leqslant f(\nu_2(B))$. If a submeasure is absolutely continuous with respect to a measure, it is exhaustive. Whether the converse is true has been a long

E-mail address: spinglass@talagrand.net (M. Talagrand).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2006.01.026

standing classical problem of measure theory. It amount to ask whether the only reason for which a submeasure can be exhaustive is because it really looks like a measure. We show that the answer to this question is negative.

Theorem 1.1. *There exists a non-zero exhaustive submeasure on the algebra $\mathcal{B}$ of clopen sets of the Cantor set that is not absolutely continuous with respect to a measure. Moreover, there is no non-zero measure on $\mathcal{B}$ that is absolutely continuous with respect to ν.*

The importance of the Control Measure Problem largely stems from the many equivalent forms that occur naturally in a variety of questions, see [2].

2. The construction

This construction elaborates on fundamental contributions by J.W. Roberts [4] and I. Farah [1].

Consider the set $T = \prod_{n\geqslant 1}\{1,\dots,2^n\}$. The algebra $\mathcal{B}$ of the clopen sets of T is isomorphic to the algebra of the clopen sets of the Cantor set $\{0,1\}^n$. We denote by $\mathcal{B}_n$ the algebra generated by the coordinates of rank $\leqslant n$. For $n \geqslant 1$, $q \leqslant 2^n$, we consider the set $S_{n,q} = \{y \in T;\ y_n \neq q\}$ and its complement $S^c_{n,q} = \{y \in T;\ y_n = q\}$. We set

$$\alpha(k) = \frac{1}{(k+5)^3}; \qquad N(k) = (k+6)^{1+(k+5)^3}.$$

There is nothing magic about these choices, what is really required is that the sequence $\alpha(k)$ decreases fast enough and the sequence $N(k)$ increases fast enough. For $k \geqslant 1$, we consider the class $\mathcal{D}_k \subset \mathcal{B}\times\mathbb{R}^+$ consisting of all couples (X,w) such that

$$X = \bigcap_{n\in I} S_{n,q(n)}, \qquad w \geqslant 2^{-k}\left(\frac{N(k)}{\operatorname{card} I}\right)^{\alpha(k)}$$

where $I \subset \mathbb{N}$, $\operatorname{card} I \leqslant N(k)$ and $q(n)$ is an arbitrary element of $\{1,\dots,2^n\}$. We set $\mathcal{D} = \bigcup_{k\geqslant 1}\mathcal{D}_k$.

For a finite subset $F = \{(A_1,w_1),\dots,(A_n,w_n)\}$ of $\mathcal{B}\times\mathbb{R}^+$, let us set

$$\cup F = \bigcup_{\ell\leqslant n} A_\ell; \qquad w(F) = \sum_{\ell\leqslant n} w_\ell.$$

Given a subset $\mathcal{C}$ of $\mathcal{B}\times\mathbb{R}^+$, we define a submeasure $\varphi_{\mathcal{C}}$ by

$$\varphi_{\mathcal{C}}(A) = \inf\bigl\{w(F);\ A \subset \cup F;\ F \subset \mathcal{C}\bigr\}.$$

Let ψ denote the submeasure $\varphi_{\mathcal{D}}$. Following Farah [1], given a submeasure θ on T, and $X \in \mathcal{B}$, we say that X is (m,n,θ)-thin if the following condition holds. For each atom A of $\mathcal{B}_m$, there exists a set $C \in \mathcal{B}_n$, $C \subset A$, $C \cap X = \varnothing$, with $\theta(\pi_A^{-1}(C)) \geqslant 1$, where π_A is the map $T \to A$ that send $\mathbf{y} = (y_i)_{i\geqslant 1}$ to $\mathbf{z} \in A$ given by $z_i = q_i$ for $i \leqslant m$, $z_i = y_i$ for $i > m$, where $q_1,\dots,q_m$ are such that $A = \{\mathbf{z} \in T;\ \forall i \leqslant m,\ z_i = q_i\}$. Given $X \in \mathcal{B}$ and a finite set I we say that X is (I,θ)-thin if it is (m,n,θ)-thin whenever $m < n$, $m,n \in I$.

Given an integer p, we proceed to the construction, by decreasing induction over k of classes $\mathcal{C}_{k,p}$ and submeasures $\varphi_{k,p}$. For $k = p$, we set $\mathcal{C}_{p,p} = \mathcal{D}$, $\varphi_{p,p} = \varphi_{\mathcal{D}} = \psi$. Having constructed $\mathcal{C}_{k+1,p}$ and $\varphi_{k+1,p}$, we set

$$\mathcal{E}_{k,p} = \left\{(X,w) \in \mathcal{B}\times\mathbb{R}^+;\ \exists I,\ X \text{ is } (I,\varphi_{k+1,p})\text{-thin},\ \operatorname{card} I \leqslant N(k),\ w \geqslant 2^{-k}\left(\frac{N(k)}{\operatorname{card} I}\right)^{\alpha(k)}\right\},$$

and we set $\mathcal{C}_{k,p} = \mathcal{E}_{k,p} \cup \mathcal{C}_{k+1,p}$, $\varphi_{k,p} = \varphi_{\mathcal{C}_{k,p}}$.

Consider an ultrafilter $\mathcal{U}$ on $\mathbb{N}$. We define the classes $\mathcal{E}_k$ by

$$(X,w) \in \mathcal{E}_k \Leftrightarrow \bigl\{p;\ (X,w) \in \mathcal{E}_{k,p}\bigr\} \in \mathcal{U}.$$

We then define the classes $\mathcal{C}_k$ by $\mathcal{C}_k = \mathcal{D} \cup \bigcup_{\ell\geqslant k}\mathcal{E}_\ell$, and the submeasures $\nu_k = \varphi_{\mathcal{C}_k}$. One can then prove the following:

Theorem 2.1. *For each k the submeasure ν_k satisfies the requirements of Theorem* 1.1.

Roughly speaking, one can say that, even though all the submeasures ν_k are exhaustive this becomes harder and harder to check as k increases. Given a disjoint sequence (E_i) of $\mathcal{B}$, it requires the same effort (as measured by how far one has to go into the sequence (E_i)) to prove that $\limsup_{i\to\infty} \nu_k(E_i) \leqslant a$ whenever $a > 2^{-k}$ than to prove that $\limsup_{i\to\infty} \nu_k(E_i) \leqslant 2^{-k}$.

The main difficulty in the proof of Theorem 2.1 is showing that the submeasures ν_k do not vanish. The main ingredient for this proof is an estimate of the type $\varphi_{k,p}(T) \geqslant C_p > 0$, which is obtained by decreasing induction over k. The proof that ν_k is exhaustive follows the lines of [1]. As for the fact that ν_k is not absolutely continuous with respect to any measure, this property is shared by any non-zero submeasure $\nu \leqslant \psi$. Indeed, such a submeasure has the property that for some $a > 0$ and n large enough, $\nu(S^c_{n,q}) \geqslant a$ for all $q \leqslant 2^n$. To see that we simply observe that for any k, any set I with card $I = N(k)$, if for $n \in I$ we choose $q(n) \leqslant 2^n$, then

$$0 < \nu(T) < \sum_{n\in I} \nu\bigl(S^c_{n,q(n)}\bigr) + \nu\left(\bigcap_{n\in I} S_{n,q(n)}\right)$$

and the last term is $\leqslant 2^{-k}$, so if $2^{-k} < \nu(T)/2$, we have $\sum_{n\in I} \nu(S^c_{n,q(n)}) \geqslant \nu(T)/2$ which suffices. Also, whenever $\nu \leqslant \psi$, there is no non-zero measure that is absolutely continuous with respect to ν, because one can show that such a property is already true for ψ.

References

[1] I. Farah, Example of ε-exhaustive pathological measures, Fund. Math. 181 (2004) 257–272.
[2] D.H. Fremlin, Measure Theory, Vol. 3: Measure Algebras, second ed., Torres Fremlin, 2004.
[3] D. Maharam, An algebraic characterization of measure algebras, Ann. Math. 48 (1947) 154–167.
[4] J.W. Roberts, Maharam's problem, in: Kranz, Labuda (Eds.), Procedings of the Orlicz Memorial Conference, 1991, unpublished.