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Partial Differential Equations

Neumann problem for a quasilinear elliptic equation in a varying domain

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Abstract

We investigate the Neumann problem for a nonlinear elliptic operator of Leray–Lions type in $\Omega^{(s)} = \Omega \backslash F^{(s)}$, $s = 1, 2, \ldots$, where Ω is a domain in $\mathbf{R}^n$ $(n \geqslant 3)$, $F^{(s)}$ is a closed set located in the neighborhood of a $(n-1)$-dimensional manifold Γ lying inside Ω. We study the asymptotic behavior of $u^{(s)}$ as $s \to \infty$, when the set $F^{(s)}$ tends to Γ. ***To cite this article: M. Sango, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Problème de Neumann pour une équation élliptique non lineaire dans un domaine perforé. Nous étudions le problème de Neumann pour un opérateur élliptique de type Leray–Lions dans un domaine $\Omega^{(s)} = \Omega \backslash F^{(s)}$, $s = 1, 2, \ldots$, où Ω est un ouvert dans $\mathbf{R}^n$ $(n \geqslant 3)$, $F^{(s)}$ est un ensemble fermé situé au voisinage d'une variété differentiable Γ de dimension $(n-1)$ à l'intérieur de Ω. Nous étudions the comportement asymptotique de $u^{(s)}$ quand $F^{(s)}$ converge vers Γ dans un sens approprié. ***Pour citer cet article : M. Sango, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Soit Ω est un ouvert dans $\mathbf{R}^n$ $(n \geqslant 3)$, $F^{(s)}$, $s = 1, 2, \ldots$, est un ensemble fermé situé au voisinage d'une variété Γ de dimension $(n-1)$ à l'intérieur de Ω qui divise Ω en deux domaines disjoints Ω^+et Ω^-. Dans le domaine perforé $\Omega^{(s)} = \Omega \backslash F^{(s)}$, nous étudions le problème aux limites

$$Au^{(s)} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_i\left(x, \frac{\partial u^{(s)}}{\partial x}\right)\right) = f, \quad \text{dans } \Omega^{(s)},$$

$$\frac{\partial u^{(s)}}{\partial \nu_A} =: \sum_{i=1}^{n} a_i\left(x, \frac{\partial u^{(s)}}{\partial x}\right)\cos(\nu, x_i) = 0, \quad \text{sur } \partial F^{(s)},$$

$$u^{(s)} = 0 \quad \text{sur } \partial\Omega,$$

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où ν est un vecteur normal à $\partial F^{(s)}$, f et A sont une fonction et un opérateur assujettis à des conditions définies dans la suite. Nous démontrons sous des hyphotèses appropriées que lorsque $s \to \infty$, la suite $u^{(s)}$ de solutions du problème converge dans des topologies convenables vers une solution du problème de transmission

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_i\left(x, \frac{\partial u}{\partial x}\right)\right) = f, \quad \text{dans } \Omega\backslash\Gamma,$$
$$\left(\frac{\partial u}{\partial \nu_A}\right)_+ + \left(\frac{\partial u}{\partial \nu_A}\right)_- = pc(x)|u_+ - u_-|^{p-2}(u_+ - u_-) \quad \text{sur } \Gamma,$$
$$u = 0 \quad \text{sur } \partial\Omega.$$

Le paramètre p et la fonction c sont définis dans la suite.

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^n$ ($n \geqslant 3$) with a sufficiently smooth boundary $\partial\Omega$. Let $F^{(s)}$ be a closed set in Ω depending on the parameter s running throughout the set of natural numbers. The main assumption on the set $F^{(s)}$ is that as $s \to \infty$, $F^{(s)}$ is located in an arbitrary small neighborhood of some smooth manifold Γ without boundary which lies inside Ω and partition Ω into two subdomains Ω^+(the interior) and Ω^-(the exterior). In the domain $\Omega^{(s)} = \Omega\backslash F^{(s)}$ that we assume sufficiently smooth, we investigate the sequence of solutions $u^{(s)}$ of the boundary value problem

$$Au^{(s)} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_i\left(x, \frac{\partial u^{(s)}}{\partial x}\right)\right) = f, \quad \text{in } \Omega^{(s)}, \tag{1}$$

$$\frac{\partial u^{(s)}}{\partial \nu_A} =: \sum_{i=1}^{n} a_i\left(x, \frac{\partial u^{(s)}}{\partial x}\right)\cos(\nu, x_i) = 0, \quad \text{on } \partial F^{(s)}, \tag{2}$$

$$u^{(s)} = 0, \quad \text{on } \partial\Omega, \tag{3}$$

where ν is a normal vector to $\partial F^{(s)}$, f is a function defined and compactly supported inside Ω (the support of f does not intersect Γ), $A : W_p^1(\mathbf{R}^n) \to W_{p'}^1(\mathbf{R}^n)$ is a monotone operator satisfying appropriate conditions.

The aim of the present Note is to investigate the behavior of the sequence $u^{(s)}$ of solutions of the problem (1)–(3). Under more precise restrictions on the set $F^{(s)}$, we show that $u^{(s)}$ converges in suitable topologies to a solution of a limit problem that we derive explicitly.

The rise of interest in Neumann problems in complicated domains in the last two decades was generated by the work of Sanchez-Palencia [9] related to perforated plane structures; commonly known now as Neumann sieve. Related works can be found in [2–4,7]. The problem (1)–(3) was originally studied by Marchenko, Khruslov and their co-workers mainly in the linear case, i.e., when a_i is independent of u (see [5]). The present work is concerned with the nonlinear case. Unlike most of the papers mentioned in the previous paragraph, the perforated domain considered here has a rather general structure.

We shall use the following well-known Lebesgue and Sobolev spaces $L_p(\cdot)$, $W_p^1(\cdot)$, $\mathring{W}_p^1(\cdot)$ ($p \geqslant 1$). We denote by $W_{p'}^{-1}(\cdot)$ the dual of $\mathring{W}_p^1(\cdot)$ where p' is the Hölder conjugate of p, i.e., $p^{-1} + p'^{-1} = 1$. If ξ is a vector we denote its Euclidean norm by $|\xi|$. We denote by C all generic constants independent of s and depending only on the data.

We assume for simplicity that $2 \leqslant p < n - 1$ and that Eq. (1) is the Euler–Lagrange equation for the functional

$$I(v) = \int_{\Omega^{(s)}} \left[A_i\left(x, \frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial x_i} - fv\right]\mathrm{d}x,$$

where the functions $A_i(x, \xi)$, $\xi = (\xi_1, \ldots, \xi_n)$ are Caratheodory and satisfy

A. for all $x \in \Omega\backslash\Omega$, $t \in \mathbf{R}$ and $\xi \in \mathbf{R}^n$, $A_i(x, t\xi) = |t|^{p-2}tA_i(x, \xi)$,

B. there exist two positive constants c_1 and c_2 such that for all $\xi, \eta \in \mathbf{R}^n$ with $\xi = (\xi_i)$, $\eta = (\eta_i)$, $i = 1, \ldots, n$,

$$\sum_{i=1}^{n}\big(A_i(x,\xi)-A_i(x,\eta)\big)(\xi_i-\eta_i)\geqslant c_1|\xi-\eta|^p, \tag{4}$$

$$\big|A_i(x,\xi)-A_i(x,\eta)\big|\leqslant c_2\big(|\xi|^{p-2}+|\eta|^{p-2}\big)|\xi-\eta|. \tag{5}$$

Therefore $a_i(x,\xi)=\sum_{k=1}^{n}\xi_k\partial A_k(x,\xi)/\partial\xi_i+A_i(x,\xi)$. Hence any minimizer of the functional I in $W_p^1(\Omega^{(s)})\cap\mathring{W}_p^1(\Omega)$ which satisfies the boundary condition (2)–(3) is a weak solution of (1)–(3), the existence of which under the above conditions is well-known.

We introduce some notations. Let γ be an arbitrary open set on Γ and let $T(\gamma,\delta)$ be a layer of thickness 2δ centered around γ. We denote by $\gamma_\delta^{\pm}$ the bases of the layer $T(\gamma,\delta)$, i.e., the surfaces located at the different sides of γ at distance δ. We set $T(\gamma,\delta,s)=T(\gamma,\delta)\backslash F^{(s)}$. Let $W(\gamma,\delta,s)=\{v\in W_p^1(T(\gamma,\delta,s))\colon\ v(x)=1 \text{ on } \gamma_\delta^+,\ v(x)=0 \text{ on } \gamma_\delta^-\}$. The main characteristic of influence of the sets $F^{(s)}$ is expressed in term of the following functions of sets

$$C_A(\gamma,\delta,s)=\inf_{\varphi^{(s)}}\int\limits_{T(\gamma,\delta,s)}\sum_{i=1}^{n}A_i\left(x,\frac{\partial\varphi^{(s)}}{\partial x}\right)\frac{\partial\varphi^{(s)}}{\partial x_i}\,\mathrm{d}x, \tag{6}$$

where infimum is taken over the functions $\varphi^{(s)}\in W(\gamma,\delta,s)$. These quantities are referred to as A-conductivity of the set $T(\gamma,\delta,s)$, following Mazya [6] where they are thoroughly investigated.

Setting $\phi(x)=u^{(s)}(x)$ in the variational formulation of problem (1)–(3) we get

$$\big\|u^{(s)}\big\|_{W_p^1(\Omega^{(s)})}\leqslant C. \tag{7}$$

We have $\Omega^{(s)}=\Omega^{(s)-}\cup\Omega^{(s)+}\cup\Gamma$, where $\Omega^{(s)\pm}=\Omega^{(s)}\cap\Omega^{\pm}$. Thus $u^{(s)}\in W_p^1(\Omega^{(s)+}\cup\Omega^{(s)-})$; i.e., that there exist the functions $u^{(s)\pm}\in W_p^1(\Omega^{(s)\pm})$ such that $u^{(s)}=(u^{(s)+},u^{(s)-})$ and $\|u\|_{W_p^1(\Omega^{(s)+}\cup\Omega^{(s)-})}=:\|u\|_{W_p^1(\Omega^{(s)+})}+\|u\|_{W_p^1(\Omega^{(s)-})}$. Analogously $W_p^1(\Omega^+\cup\Omega^-)=:W_p^1(\Omega^+)\times W_p^1(\Omega^-)$ with the norm $\|u\|_{W_p^1(\Omega^+\cup\Omega^-)}=:\|u\|_{W_p^1(\Omega^+)}+\|u\|_{W_p^1(\Omega^-)}$.

We make the following hypothesis: The domains $\Omega^{(s)\pm}$ are such that for all s there exists a uniformly bounded extension operator from $W_p^1(\Omega^{(s)+}\cup\Omega^{(s)-})$ into $W_p^1(\Omega^+\cup\Omega^-)$. In the sequel a function $u^{(s)}$ in $W_p^1(\Omega^{(s)+}\cup\Omega^{(s)-})$ and its extension in $W_p^1(\Omega^+\cup\Omega^-)$ will be denoted by the same symbol.

2. Main result

The main result of this Note is:

Theorem 1. *Assume that the above conditions on problem* (1)–(3) *are satisfied and $f\in W_{p'}^{-1}(\Omega\backslash\Gamma)$. As $s\to\infty$, we require that*

(a) *the set $F^{(s)}$ lies in an arbitrary small neighborhood of the manifold $\Gamma\subset\Omega$,*
(b) *for any portion $\gamma\in\Gamma$, there exist the limits*

$$\lim_{\delta\to\infty}\underline{\lim}_{s\to\infty}C_A(\gamma,\delta,s)=\lim_{\delta\to\infty}\overline{\lim}_{s\to\infty}C_A(\gamma,\delta,s)=\int\limits_{\gamma}c(x)\,\mathrm{d}\Gamma, \tag{8}$$

where c is a nonnegative, measurable function on Γ.

Then the sequence of solutions $u^{(s)}$ of problem (1)–(3) *converges weakly in $W_p^1(\Omega^+\cup\Omega^-)$ and strongly in $W_q^1(\Omega^+\cup\Omega^-)$, $1<q<p$, to a function u which is a solution of the transmission problem*

$$-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(a_i\left(x,\frac{\partial u}{\partial x}\right)\right)=f,\quad \text{in } \Omega\backslash\Gamma, \tag{9}$$

$$\left(\frac{\partial u}{\partial\nu_A}\right)_+ +\left(\frac{\partial u}{\partial\nu_A}\right)_- =pc(x)|u_+-u_-|^{p-2}(u_+-u_-),\quad \text{on } \Gamma, \tag{10}$$

$$u=0,\quad \text{on } \partial\Omega, \tag{11}$$

where the signs + and − indicate the boundary values of the function on the different sides of Γ, $(\frac{\partial u}{\partial \nu_A})_\pm$ is the derivative along the normal to Γ in the direction corresponding to ±.

Let $T(\Gamma,\delta)$ be a layer of thickness 2δ centered around the manifold Γ. Let $T(\Gamma,\delta,s)=T(\Gamma,\delta)\backslash F^{(s)}$. We consider the functional

$$\Phi_\delta^{(s)}\big(\psi^{(s)}\big)=\int\limits_{T(\Gamma,\delta,s)}\sum_{i=1}^n A_i\left(x,\frac{\partial\psi^{(s)}}{\partial x}\right)\frac{\partial\psi^{(s)}}{\partial x_i}\,\mathrm{d}x,$$

over the set $\widetilde{W}$ of functions from $W_p^1(T(\Gamma,\delta,s))$ taking on the surfaces Γ_δ^+, Γ_δ^- bounding the layer $T(\Gamma,\delta)$ the values of $u(x)\in W_p^1(\Omega^+\cup\Omega^-)$. It is a well known fact that under the growth conditions on A_i, there exists at least a function $u^{(s)}$ minimizing $\Phi_\delta^{(s)}$, i.e.,

$$\Phi_\delta^{(s)}\big(u^{(s)}\big)=\inf_{\psi^{(s)}\in\widetilde{W}}\Phi_\delta^{(s)}\big(\psi^{(s)}\big).$$

The following key result holds:

Theorem 2. *Assume that the conditions of Theorem 1 are satisfied. Then for any function $u\in W_p^1(\Omega^+\cup\Omega^-)$ the following relation holds*

$$\lim_{\delta\to 0}\overline{\lim}_{s\to\infty}\Phi_\delta^{(s)}(u)=\lim_{\delta\to 0}\underline{\lim}_{s\to\infty}\Phi_\delta^{(s)}(u)=\int\limits_\Gamma c|u^+-u^-|^p\,\mathrm{d}\Gamma.$$

This theorem gives an accurate behavior of the energy in the vinicity of the sets $F^{(s)}$ and is responsible for the appearance of the additional term in the transmission conditions.

3. Proof of Theorem 1

We give an idea of the proof of Theorem 1. From (7) and the existence of the extension assumed in the theorem it follows that $\|u^{(s)}\|_{W_p^1(\Omega^+\cup\Omega^-)}\leqslant C$. Therefore a function $u\in W_p^1(\Omega^+\cup\Omega^-)$ exists such that $u^{(s)}$ converges to $u$weakly in $W_p^1(\Omega^+\cup\Omega^-)$. In fact following the arguments of Boccardo and Murat [1] we get a more precise convergence, namely $u^{(s)}$ strongly converges to u in $W_q^1(\Omega^+\cup\Omega^-)$, $1<q<p$. Let $u^\pm$ be the restriction of u to $\Omega_\delta^\pm$. We show that $u^\pm$ satisfies the relation

$$\int\limits_{\Omega_\delta^\pm}\sum_{i=1}^n A_i\left(x,\frac{\partial u^\pm}{\partial x}\right)\frac{\partial\varphi}{\partial x_i}\,\mathrm{d}x=\int\limits_{\Omega_\delta^\pm}f\varphi\,\mathrm{d}x,\quad\forall\varphi\in\mathring{W}_p^1\big(\Omega_\delta^\pm\big).$$

Using this relation together with the conditions (4)–(5) on $A(x,\xi)$ and some appropriate estimates we get that

$$u^{(s)}\to u^\pm,\quad\text{strongly in }\mathring{W}_p^1\big(\Omega_\delta^\pm\big).\tag{12}$$

Next we let $w\in\mathring{W}_p^1(\Omega^+\cup\Omega^-)$ be arbitrary and define the function $w_\delta^{(s)}$ by: $w_\delta^{(s)}(x)=w(x)$ if $x\in\Omega_\delta^\pm$ and $w_\delta^{(s)}(x)=w^{(s)}(x)$ if $x\in T(\Gamma,\delta)$, where $w^{(s)}\in W_p^1(\Omega^{(s)})$ and is a minimizer of $\Phi_\delta^{(s)}$ in $W_p^1(T(\Gamma,\delta,s))$. Let

$$J(w)=\int\limits_{\Omega^+\cup\Omega^-}\left[\sum_{i=1}^n A_i\left(x,\frac{\partial w}{\partial x}\right)\frac{\partial w}{\partial x_i}+fw\right]\mathrm{d}x+\int\limits_\Gamma c(x)|w^+-w^-|^p\,\mathrm{d}\Gamma,\tag{13}$$

be a functional on $\overset{\circ}{W}{}^1_p(\Omega^+\cup\Omega^-)$, the class of functions in $W^1_p(\Omega^+\cup\Omega^-)$ which vanish on $\partial\Omega$. Under the conditions imposed on the functions $A_i(x,p)$, $(x,p)\in\mathbf{R}^{2n}$, any minimizer of the functional J in $\overset{\circ}{W}{}^1_p(\Omega^+\cup\Omega^-)$ is also a weak solution of problem (9)–(11). We prove that the function u minimizes J in $\overset{\circ}{W}{}^1_p(\Omega^+\cup\Omega^-)$. We have

$$I\big(w_\delta^{(s)}\big)=\int\limits_{\Omega_\delta^+\cup\Omega_\delta^-}\left[\sum_{i=1}^n A_i\left(x,\frac{\partial w}{\partial x}\right)\frac{\partial w}{\partial x_i}+fw\right]+\Phi_\delta^{(s)}(w).$$

By Theorem 2 we get

$$\lim_{\delta\to 0}\overline{\lim}_{s\to\infty} I\big(w_\delta^{(s)}\big)=J(w).\tag{14}$$

Next let $u_\delta^{(s)}\in\overset{\circ}{W}{}^1_p(\Omega^+\cup\Omega^-)$ be an extension of $u^{(s)}$ from $\Omega_\delta^+\cup\Omega_\delta^-$ to $\Omega^+\cup\Omega^-$ such that $u_\delta^{(s)}\to u$ strongly in $\overset{\circ}{W}{}^1_p(\Omega^+\cup\Omega^-)$ as $\delta\to 0$, $s\to\infty$. We have

$$I\big(u^{(s)}\big)=\int\limits_{\Omega_\delta^+\cup\Omega_\delta^-}\left[\sum_{i=1}^n A_i\left(x,\frac{\partial u_\delta^{(s)}}{\partial x}\right)\frac{\partial u_\delta^{(s)}}{\partial x_i}+fu_\delta^{(s)}\right]+\Phi_\delta^{(s)}\big(u_\delta^{(s)}\big).$$

By Theorem 2 and estimates involving (4), (5) and (12) we get

$$\lim_{\delta\to 0}\underline{\lim}_{s\to\infty} I\big(u^{(s)}\big)\geqslant J(u).\tag{15}$$

We have $I(u^{(s)})\leqslant I(w_\delta^{(s)})$. Thus (14) and (15) imply that $J(u)\leqslant J(w)$. w being arbitrary we get that u minimizes J and therefore satisfies (9)–(11).

Next we give an example of a geometry for $F^{(s)}$ for which the function c in (8) can be explicitly computed. In $\mathbf{R}^n$, we consider for each s a layer $T^{(s)}$ of thickness $h^{(s)}$ bounded from one side by a sphere Γ and from the other side by another sphere $\Gamma^{(s)}$ parallel to Γ and at a distance $h^{(s)}$ from it. We remove from Γ s disjoint connected open sets $\sigma_i=\sigma_i^{(s)}$ of diameter $d_i^{(s)}$. The normals through the points $x\in\sigma_i$, cut some channels $T_i^{(s)}$ through $T^{(s)}$. Set $F^{(s)}=\overline{T^{(s)}\setminus\bigcup_{i=1}^s T_i^{(s)}}$. Let Ω be smooth bounded domain in $\mathbf{R}^n$ containing $\overline{T^{(s)}}$. In the region $\Omega^{(s)}=\Omega\setminus F^{(s)}$, we consider the boundary value problem

$$\Delta_p u^{(s)}=-\sum_{j=1}^n\frac{\partial}{\partial x_j}\left(\left|\frac{\partial u^{(s)}}{\partial x}\right|^{p-2}\frac{\partial u^{(s)}}{\partial x_j}\right)=f,\quad x\in\Omega^{(s)},\tag{16}$$

$$\frac{\partial u^{(s)}}{\partial\nu_{\Delta_p}}=0,\quad\text{on }\partial F^{(s)},\qquad u=0\quad\text{on }\partial\Omega.\tag{17}$$

We denote by Ω^+ (Ω^-) the region interior (exterior) to Γ and by $\Omega^{(s)-}$ the set $\Omega^{(s)}\setminus\overline{\Omega^-}$. Through appropriate rescalings the arguments of [8, §4] related to the construction of extension operator for perforated domains of type I can be used to produce an extension from $W^1_p(\Omega^{(s)-})$ into $W^1_p(\Omega^-)$ uniformly bounded. Hence the required extension from $W^1_p(\Omega^{(s)+}\cup\Omega^{(s)-})$ into $W^1_p(\Omega^+\cup\Omega^-)$ follows. Let γ be a portion of the surface Γ and $T(\gamma,\delta)$ be the layer with thickness 2δ centered around γ with bases $\gamma_\delta^\pm$. We define the quantity

$$C_{\Delta_p}(\gamma,\delta,s)=\frac{1}{p}\inf_{w^{(s)}}\int\limits_{T(\gamma,\delta,s)}\left|\frac{\partial w^{(s)}}{\partial x}\right|^p\mathrm{d}x,\tag{18}$$

where $T(\gamma,\delta,s)=T(\gamma,\delta)\setminus\overline{T^{(s)}\setminus\bigcup_{i=1}^s T_i^{(s)}}$, and the infimum is taken over the functions $w^{(s)}\in W(\gamma,\delta,s)$. We denote $\overline{T_i^{(s)}}\cap\Gamma$ and $\overline{T_i^{(s)}}\cap\Gamma^{(s)}$ by $\sigma_i^{(s)-}$ and $\sigma_i^{(s)+}$, respectively. Let $R_i^{(s)}$ be the distance between $\sigma_i^{(s)-}$ and $\bigcup_{i\neq j}\sigma_j^{(s)-}$ and assume $\max_i\{R_i^{(s)},d_i^{(s)}\}<\delta$, for all s. We make the following assumptions:

$$\overline{\lim_{s\to\infty}}\sum_{\gamma(s)}\frac{[d_i^{(s)}]^{n-1}}{[h^{(s)}R_i^{(s)}]^p}\leqslant C_2;\quad \text{as } s\to\infty R_i^{(s)}=\mathrm{o}\big(d_i^{(s)}\big)\to 0;$$
$$\overline{\lim_{s\to\infty}}\sum_{\gamma(s)}\mathrm{mes}\big(\sigma_i^{(s)}\big)\big[h^{(s)}\big]^{1-p}=\int_\gamma c(x)\,\mathrm{d}\Gamma, \tag{19}$$

where $c(x)$ is a nonnegative function on Γ, $\sum_{\gamma(s)}$ is the sum over all i for which $\sigma_i^{(s)}$ belong to $\gamma\subset\Gamma$ and C_i are positive constants. Then we have the following result

Theorem 3. *Let the conditions* (19) *be satisfied and* $n>p+1$, *then the sequence of solutions* $u^{(s)}\in W_p^1(\Omega^{(s)})$ *of problem* (16)–(17) *converges weakly in* $W_p^1(\Omega^+\cup\Omega^-)$ *to a function* $u(x)$ *which is a solution of the problem*

$$\Delta_p u=f,\quad \text{in } \Omega\backslash\Gamma, \tag{20}$$
$$\left(\frac{\partial u}{\partial\nu_{\Delta_p}}\right)_+ +\left(\frac{\partial u}{\partial\nu_{\Delta_p}}\right)_- = pc(x)|u_+-u_-|^{p-2}(u_+-u_-)\quad \text{on } \Gamma,\qquad u=0\quad \text{on } \partial\Omega, \tag{21}$$

where c *is the function defined in* (19).

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