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Partial Differential Equations

Dispersion and Strichartz estimates for the Liouville equation

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Abstract

We consider the Liouville equation associated to a metric g and we prove dispersion and Strichartz estimates for the solution of this equation in terms of the geometry of the trajectories associated to g. In particular, we obtain global Strichartz estimates in time for metrics where dispersion estimate is false even locally in time. We also study the analogy between Strichartz estimates obtained for the Liouville equation and the Schrödinger equation with variable coefficients. ***To cite this article: D. Salort, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Dispersion et estimations de Strichartz pour l'équation de Liouville. On considère l'équation de Liouville associée à une métrique g et on prouve des estimations de dispersion et de Strichartz pour la solution de cette équation en fonction de la géométrie des trajectoires associée à g. En particulier, on obtient des estimations de Strichartz globales en temps pour des métriques où l'estimation de dispersion est fausse même pour des temps arbitrairement petits. Cette étude permet de mettre en évidence une analogie entre le comportement de la solution de l'équation de Schrödinger et de l'équation de Liouville. ***Pour citer cet article : D. Salort, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

The aim of this Note is on the one hand to understand the link between the geometry of the trajectories and dispersion phenomena and on the other hand to establish Strichartz estimates in a very general setting (see [6] for more details). We consider here the d-dimensional case with $d \geqslant 2$, (see [7] for the one dimensional case) and a $\mathcal{C}^2$ metric $g : \mathbb{R}^d \to M_d(\mathbb{R})$ satisfying the following condition

$$\exists m > 0,\ \exists M > 0, \quad m\,\mathrm{Id} \leqslant g \leqslant M\,\mathrm{Id}.$$

We denote $g^{ij} = (g^{-1})_{ij}$ and we consider the Liouville equation associated to a metric g given by

$$\begin{cases} \partial_t f + \{H, f\} = 0, \\ f(0, x, \xi) = f^{\mathrm{in}}(x, \xi), \end{cases} \tag{1}$$

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where $\{H, f\}$ denotes the Poisson bracket between f and the Hamiltonian $H(x,\xi) := \frac{1}{2}\sum_{i,j=1}^{d} g^{ij}(x)\xi_i\xi_j$. The solution $f(t,x,\xi)$ of this equation models the evolution of a microscopic density of particles which at time t, is at position x, and velocity ξ.

Definition 1.1. Let $(q,p,r,a) \in [1,+\infty]^4$. We say that the quadruplet (q,p,r,a) is admissible if it satisfies the following relations

$$\frac{2}{q} = d\left(\frac{1}{r} - \frac{1}{p}\right), \quad \frac{1}{a} = \frac{1}{2}\left(\frac{1}{r} + \frac{1}{p}\right) \quad \text{and} \quad q > 2 \geqslant a.$$

In the case where $g = \mathrm{Id}$, F. Castella and B. Perthame have shown in [4] that the solution of the Liouville equation satisfies

$$\|f\|_{L_t^q(L_x^p(L_\xi^r))} \leqslant C(q,p,r,a)\|f^{\mathrm{in}}\|_{L_{x,\xi}^a}$$

for all admissible quadruplets (q,p,r,a).

2. Estimations without geometrical assumptions

In this section, we consider the case of general bounded positive metrics without geometric assumptions and we obtain the following theorem

Theorem 2.1. *Let I be a finite time interval, (q,p,r,a) an admissible quadruplet and $\epsilon > 0$. Then there exists a constant C such that, the solution f of Eq.* (1)*, satisfies*

$$\|f\|_{L_I^q(L_x^p(L_\xi^r))} \leqslant C\left\|\left(1 + |\xi|^{1/q+\epsilon}\right)f^{\mathrm{in}}\right\|_{L_{x,\xi}^a}.$$

Moreover, the loss of $\frac{1}{q}$ momentum is optimal for the Liouville equation on $M \times TM$ for all Riemannian compact manifold M.

The idea of the proof is to establish a dispersion estimate both local in time and frequency and to make a time-frequency split-up of the solution. This split-up method has been used for wave equations by H. Bahouri and J.-Y. Chemin [1] and in the case of Schrödinger equation by N. Burq, P. Gérard, N. Tzvetkov [3] and by the author [8]. To prove that the loss of $\frac{1}{q}$ momentum is optimal, we consider the sequence of initial data $f_n^{\mathrm{in}}(x,\xi) = \mathbb{I}_M(x)\mathbb{I}_{2^n \leqslant |\xi| \leqslant 2^{n+1}}(\xi)$ where $n \in \mathbb{N}$ and we use the fact that $\frac{d}{r} = \frac{d}{a} + \frac{1}{q}$.

3. Dispersive metrics

In this section, we give a condition on the trajectories for the solution f of the Liouville equation (1) to satisfy the same dispersion estimate as in the Euclidean case. To do this, we introduce the following definitions

Definition 3.1. Let $x \in \mathbb{R}^d$, we denote $\mathbb{S}_x^{d-1} := \{\xi \in \mathbb{R}^d,\ \|\xi\|_x = (\sum_{i,j} g^{ij}(x)\xi_i\xi_j)^{1/2} = 1\}$. We denote by σ_x the surface measure of $\mathbb{S}_x^{d-1}$ and σ_e the surface measure of the Euclidean sphere $\mathbb{S}^{d-1}$. We say that a metric g is non-focusing if

$$\exists C,\ \forall (t,x,y) \in \mathbb{R}^{2d+1},\ \forall \epsilon > 0, \quad \sigma_x\left\{e \in \mathbb{S}_x^{d-1},\ X(t,x,e) \in B(y,\epsilon)\right\} \leqslant C\left(\frac{\epsilon}{|t|}\right)^{d-1} \tag{2}$$

i.e. if the trajectories do not concentrate at any point and scatter in all directions. We say that g is focusing if it does not satisfy the condition (2).

Definition 3.2. Let g be a metric of $\mathbb{R}^d$. We say that g is dispersive if g is non-focusing and further satisfies the condition

$$\exists C,\ \forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d,\ \forall \epsilon > 0, \quad \mu\left\{t \in \mathbb{R},\ \exists e \in \mathbb{S}_x^{d-1},\ X(\cdot,x,e) \in B(y,\epsilon)\right\} \leqslant C\epsilon, \tag{3}$$

where μ denotes the Lebesgue measure on the real line.

We obtain the following theorems:

Theorem 3.3. *Let g be a dispersive metric and let f be the solution of the Liouville equation* (1). *Then, we have the following dispersion estimate*

$$\sup_x \left(\int \left| f(t,x,\xi) \right| \mathrm{d}\xi \right) \leqslant C|t|^{-d} \int \sup_\xi \left(\left| f^{\mathrm{in}} \right|(x,\xi) \right) \mathrm{d}x.$$

Theorem 3.4. *Let g be a focusing metric. Then there exists a sequence $(f_n^{\mathrm{in}})_{n\in\mathbb{N}^*}$ of initial data, a sequence of reals $(t_n)_{n\in\mathbb{N}^*}$, such that the sequence of solutions $(f_n)_{n\in\mathbb{N}^*}$ of Eq.* (1) *with initial data $(f_n^{\mathrm{in}})_{n\in\mathbb{N}^*}$ satisfies*

$$\sup_x \int_\xi \left| f_n(t_n,x,\xi) \right| \mathrm{d}\xi \geqslant C \frac{n}{|t_n|^d} \int \sup_\xi \left| f_n^{\mathrm{in}}(x,\xi) \right| \mathrm{d}x.$$

To prove Theorem 3.3, we first consider the particular case where $f^{\mathrm{in}}(x,\xi) = \mathbb{I}_{B(\alpha_1,r_1)}(x)$ with $\alpha_1 \in \mathbb{R}^d$ and $r_1 \in \mathbb{R}^+$ and we then proceed in the general case by an approximation argument. To prove Theorem 3.4, we consider initial data f_n^{in} which concentrated in space around a point where the metric focalized.

4. Case of non-trapped and long range perturbation of the Euclidean metric

For the definition of non-trapped and long range perturbation of the Euclidean metric, we refer to the article of N. Burq [2]. The following proposition shows that the set of dispersive metrics does not contain the set of non-trapped metrics which coincide with the Euclidean metric outside a compact subset of $\mathbb{R}^2$.

Proposition 4.1. *Let $\psi \in \mathcal{C}^\infty(\mathbb{R})$ be a non-negative function such that*

$$\psi \equiv 1 \quad \text{on }]-\infty, R] \quad \text{and} \quad \psi \equiv 0 \quad \text{on } [2R, +\infty[,$$

where R is a constant large enough. Let S be the surface of $\mathbb{R}^3$ given by

$$S = \left\{ (x,y,z) \in \mathbb{R}^3,\ z = \frac{1}{2} r^2 \psi(r),\ \text{where } r = \sqrt{x^2+y^2} \right\}.$$

Let us denote by ϕ the diffeomorphism from $\mathbb{R}^2$ to S given by $\phi(x,y) = (x,y,\frac{1}{2}r^2\psi(r))$. Let $g : \mathbb{R}^2 \to M_2(\mathbb{R})$ be the metric given by

$$g_{ij}(x) = \left(\frac{\partial \phi}{\partial x_i}(x) \,\middle|\, \frac{\partial \phi}{\partial x_j}(x) \right)_{\mathbb{R}^3}.$$

Then, g is a non-trapped compactly supported perturbation of the Euclidean metric and non-focusing and so non-dispersive.

The proof of this proposition is given by using the equations on geodesics and the fact that the paraboloid has two conjugate points.

The following proposition shows an averaging effect for the solution of the Liouville equation (1) associated to a non-trapped long range perturbation of the Euclidean metric. In the case of the Liouville equation associated to the Euclidean metric, P.-L. Lions and B. Perthame [5] have shown other averaging lemma in a different setting.

Proposition 4.2. *Let g be a non-trapped and long range perturbation of the Euclidean metric. Let $R > 0$, and let γ be a function in $\mathcal{D}(B(0,R))$. Then, there exists $C > 0$ such that the solution f of the Liouville equation* (1) *satisfies for all $a \geqslant 1$,*

$$\left\| |\xi|^{1/a} \gamma(x) f(t,x,\xi) \right\|_{L^a_{t,x,\xi}} \leqslant C \left\| f^{\mathrm{in}} \right\|_{L^a_{x,\xi}}.$$

This proposition and the study made in the case without geometric assumptions allow us to prove Theorems 4.3 and 4.4 via a split-up in time, space and frequency of the solution. This strategy of splitting-up the solution has been used for the Schrödinger equation by N. Burq in [2].

We have the following theorems:

Theorem 4.3. *Let g be a non-trapped metric and let $R > 0$ such that g coincides with the Euclidean metric on ${}^{c}B(0, R)$. Then, there exists a constant C such that the solution f of the Liouville equation* (1) *satisfies*

$$\|f\|_{L_t^q(L_x^p(L_\xi^r))} \leqslant C\bigg(\sum_{j\in\mathbb{Z}} \big\|\varphi\big(2^{-j}\xi\big) f^{\mathrm{in}}\big\|_{L_{x,\xi}^a}^r\bigg)^{1/r} \quad \textit{where } \sum_{j\in\mathbb{Z}} \varphi\big(2^{-j}\xi\big) = 1\ \forall \xi \in \mathbb{R}^d \setminus \{0\}$$

for all admissible quadruplets (q, p, r, a).

This theorem gives a global Strichartz estimate in time. Nevertheless, the average effect giving poor control on low frequencies, the norm in which the initial data is considered is very slightly weakened compared to the case of the Euclidean metric. More generally, we have the following theorem

Theorem 4.4. *Let g be a non-trapped metric which is a long range perturbation of the Euclidean metric. Let f be the solution of the Liouville equation* (1)*. Then, for all $\epsilon > 0$, for all finite interval I, there exists a constant C such that we have*

$$\|f\|_{L_I^q(L_x^p(L_\xi^r))} \leqslant C\big\|\big(1+|\xi|\big)^{\epsilon} f^{\mathrm{in}}\big\|_{L_{x,\xi}^a},$$

where (q, p, r, a) is an admissible quadruplet.

References

[1] H. Bahouri, J.-Y. Chemin, Équations d'ondes quasilinéaires et estimations de Strichartz, Amer. J. Math. 121 (6) (1999) 1337–1377.

[2] N. Burq, Estimations de Strichartz pour des perturbations à longue portée de l'opérateur de Schrödinger, Séminaire EDP de l'Ecole polytechnique.

[3] N. Burq, P. Gérard, N. Tzvetkov, Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds, Amer. J. Math. 126 (3) (2004) 569–605.

[4] F. Castella, B. Perthame, Estimations de Strichartz pour les équations de transport cinétique, C. R. Acad. Sci. Paris Sér. I Math. 322 (6) (1996) 535–540.

[5] P.-L. Lions, B. Perthame, Lemmes de moments, de moyenne et de dispersion, C. R. Acad. Sci. Paris Sér. I Math. 314 (11) (1992) 801–806.

[6] D. Salort, Dispersion and Strichartz estimates for the Liouville equation, submitted for publication.

[7] D. Salort, Weighted dispersion and Strichartz estimates for the Liouville equation in one dimension, submitted for publication.

[8] D. Salort, The Schrödinger equation type with a nonelliptic operator, submitted for publication.