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Algebraic Geometry/Group Theory

Extended Picard complexes for algebraic groups and homogeneous spaces

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Abstract

For a smooth geometrically integral algebraic variety X over a field k of characteristic 0, we define the extended Picard complex $\mathrm{UPic}(\overline{X})$. It is a complex of length 2 which combines the Picard group $\mathrm{Pic}(\overline{X})$ and the group $U(\overline{X}) := \bar{k}[\overline{X}]^\times/\bar{k}^\times$, where $\bar{k}$ is a fixed algebraic closure of k and $\overline{X} = X \times_k \bar{k}$. For a connected linear k-group G we compute the complex $\mathrm{UPic}(\overline{G})$ (up to a quasi-isomorphism) in terms of the algebraic fundamental group $\pi_1(\overline{G})$. We obtain similar results for a homogeneous space X of a connected k-group G. ***To cite this article: M. Borovoi, J. van Hamel, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Complexes de Picard étendus pour des groupes algébriques et des espaces homogènes. Soient k un corps de caractéristique zéro et X une k-variété algébrique lisse et géométriquement intègre. Nous définissons le complexe de Picard étendu $\mathrm{UPic}(\overline{X})$. C'est un complexe de longueur 2 qui combine le groupe de Picard $\mathrm{Pic}(\overline{X})$ et le groupe $U(\overline{X}) := \bar{k}[\overline{X}]^\times/\bar{k}^\times$, où $\bar{k}$ est une clôture algébrique fixée de k et $\overline{X} = X \times_k \bar{k}$. Pour un k-groupe linéaire connexe G, nous calculons le complexe $\mathrm{UPic}(\overline{G})$ (à quasi-isomorphisme près) en termes du groupe fondamental algébrique $\pi_1(\overline{G})$. Nous obtenons des résultats similaires pour un espace homogène X d'un k-groupe connexe G. ***Pour citer cet article : M. Borovoi, J. van Hamel, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Throughout the Note, k denotes a field of characteristic 0 and $\bar{k}$ is a fixed algebraic closure of k. By a k-group we mean a linear algebraic group defined over k.

Let G be a connected reductive k-group. Let

$$\rho : G^{\mathrm{sc}} \twoheadrightarrow G^{\mathrm{ss}} \hookrightarrow G$$

be Deligne's homomorphism, where G^{ss} is the derived subgroup of G (it is semisimple) and G^{sc} is the universal covering of G^{ss} (it is simply connected). Let $T \subset G$ be a maximal torus (defined over k) and let $T^{\mathrm{sc}} := \rho^{-1}(T)$ be the

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corresponding maximal torus of G^{sc}. The 2-term complex of tori

$$T^{\mathrm{sc}} \xrightarrow{\rho} T$$

(with T^{sc} in degree -1) plays an important role in the study of the arithmetic of reductive groups. For example, the Galois hypercohomology $H^i(k, T^{\mathrm{sc}} \to T)$ of this complex is the abelian Galois cohomology of G (cf. [1]). The corresponding Galois module

$$\mathbf{X}_*(\overline{T})/\rho_*\mathbf{X}_*(\overline{T}^{\mathrm{sc}})$$

(where $\mathbf{X}_*$ denotes the cocharacter group of a torus) is called the algebraic fundamental group $\pi_1(\overline{G})$ (loc. cit.). The related complex group with holomorphic $\mathrm{Gal}(\bar{k}/k)$-action

$$\mathrm{Hom}(\pi_1(\overline{G}), \mathbf{C}^\times) = \ker(\mathbf{X}^*(T) \otimes \mathbf{C}^\times \to \mathbf{X}^*(T^{\mathrm{sc}}) \otimes \mathbf{C}^\times)$$

(where $\mathbf{X}^*$ denotes the character group of an algebraic group) is canonically isomorphic to the center of a connected Langlands dual group $\widehat{G}$ for G, considered by Kottwitz [7].

Clearly, the above constructions rely on the linear algebraic group structure of $\overline{G}$. However we show in this note that they are related to a very natural geometric/cohomological construction that works for an arbitrary smooth k-variety X. The proofs will be published elsewhere.

1. The extended Picard complex

By a k-variety we mean a smooth geometrically integral k-variety. If X is a k-variety, we write $\overline{X}$ for $X \times_k \bar{k}$. We write $\bar{k}[\overline{X}]$ (resp. $\bar{k}(\overline{X})$) for the ring of regular functions (resp. the field of rational functions) on $\overline{X}$.

For a k-variety X, consider the cone $\mathrm{UPic}(\overline{X})$ of the morphism

$$\mathbf{G}_{\mathrm{m}}(\bar{k}) \to \tau_{\leqslant 1} R\Gamma(\overline{X}, \mathbf{G}_{\mathrm{m}})$$

in the derived category of discrete Galois modules. More explicitly, this cone is represented by the 2-term complex

$$\bar{k}(X)^\times/\bar{k}^\times \to \mathrm{Div}(\overline{X})$$

(with $\bar{k}(X)^\times/\bar{k}^\times$ in degree 0), where Div denotes the divisor group. It follows from the definitions that the cohomology groups $\mathcal{H}^i$ of the complex $\mathrm{UPic}(\overline{X})$ vanish for $i \neq 0, 1$, and

$$\mathcal{H}^0(\mathrm{UPic}(\overline{X})) = U(\overline{X}) := \bar{k}[\overline{X}]^\times/\bar{k}^\times, \qquad \mathcal{H}^1(\mathrm{UPic}(\overline{X})) = \mathrm{Pic}(\overline{X}).$$

Hence $\mathrm{UPic}(\overline{X})$ can be regarded as a 2-extension of $\mathrm{Pic}(\overline{X})$ by $U(\overline{X})$. We shall call this complex the *extended Picard complex* of X.

Lemma 1.1. *Let X_c be a smooth compactification of a k-variety X. Then there is a distinguished triangle*

$$\mathrm{UPic}(\overline{X}) \to \mathrm{Div}_{\overline{X}_c \setminus \overline{X}}(\overline{X}) \to \mathrm{Pic}(\overline{X}_c) \to \mathrm{UPic}(\overline{X})[1]$$

where $\mathrm{Div}_{\overline{X}_c \setminus \overline{X}}(\overline{X})$ *is the permutation module of divisors in the complement of $\overline{X}$ in $\overline{X}_c$.*

Now we consider $\mathrm{Pic}(X) = H^1(X, \mathbf{G}_{\mathrm{m}})$ and $\mathrm{Br}(X) = H^2_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$ (over k). Consider the canonical homomorphisms $\mathrm{Br}(k) \xrightarrow{\alpha} \mathrm{Br}(X) \xrightarrow{\beta} \mathrm{Br}(\overline{X})$ and set $\mathrm{Br}_{\mathrm{a}}(X) = \ker\beta/\operatorname{im}\alpha$.

Lemma 1.2. *Let X be a k-variety.*

(i) *There is a natural injection* $\mathrm{Pic}(X) \hookrightarrow H^1(k, \mathrm{UPic}(\overline{X}))$, *which is an isomorphism if $X(k) \neq \emptyset$.*
(ii) *There is a natural injection* $\mathrm{Br}_{\mathrm{a}}(X) \hookrightarrow H^2(k, \mathrm{UPic}(\overline{X}))$, *which is an isomorphism if $X(k) \neq \emptyset$ or if $H^3(k, \mathbf{G}_{\mathrm{m}}) = 0$ (e.g. when k is a number field).*

If C is a complex of $\mathrm{Gal}(\bar{k}/k)$-modules, we write $Ш^i_\omega(k, C) = \ker[H^i(k, C) \to \prod_\gamma H^i(\gamma, C)]$ where γ runs over all closed procyclic subgroups of $\mathrm{Gal}(\bar{k}/k)$.

Proposition 1.3. *Let X_c be a smooth compactification of a smooth k-variety X. The triangle of Lemma* 1.1 *gives rise to an isomorphism*

$$\text{Ш}^1_\omega\big(k, \operatorname{Pic}\big(\overline{X}_c\big)\big) \xrightarrow{\sim} \text{Ш}^2_\omega\big(k, \operatorname{UPic}\big(\overline{X}\big)\big).$$

This is particularly interesting for a homogeneous variety X of a connected k-group G with connected geometric stabilizer, for which we have $\text{Ш}^1_\omega(k, \operatorname{Pic}(\overline{X}_c)) = H^1(k, \operatorname{Pic}(\overline{X}_c))$, see [4].

2. Algebraic groups and torsors

Let G be a connected reductive k-group. We define the dual complex $\pi_1(\overline{G})^D$ to $\pi_1(\overline{G})$ by

$$\pi_1\big(\overline{G}\big)^D = \big(\mathbf{X}^*\big(\overline{T}\big) \to \mathbf{X}^*\big(\overline{T}^{\mathrm{sc}}\big)\big) \quad (\text{with } \mathbf{X}^*\big(\overline{T}\big) \text{ in degree } 0).$$

Theorem 2.1. *For a connected reductive k-group G there is a canonical, functorial in G isomorphism* (*in the derived category of discrete Galois modules*)

$$\operatorname{UPic}\big(\overline{G}\big) \xrightarrow{\sim} \pi_1\big(\overline{G}\big)^D.$$

Let G be any connected linear k-group, not necessarily reductive. We write G^{u} for the unipotent radical of G, and set $G^{\mathrm{red}} = G/G^{\mathrm{u}}$ (it is reductive). We define $\pi_1(\overline{G}) := \pi_1(\overline{G}^{\mathrm{red}})$.

Corollary 2.2. *For any connected linear k-group G we have a canonical isomorphism* $\operatorname{UPic}(\overline{G}) \xrightarrow{\sim} \pi_1(\overline{G})^D$.

Combining Corollary 2.2 with Lemma 1.2, we find a new proof of the following result.

Corollary 2.3 *(Kottwitz [7]). For any connected linear k-group G we have canonical isomorphisms* $\operatorname{Pic}(G) \xrightarrow{\sim} H^1(k, \pi_1(\overline{G})^D)$ *and* $\operatorname{Br_a}(G) \xrightarrow{\sim} H^2(k, \pi_1(\overline{G})^D)$.

Theorem 2.1 gives a description of the complex UPic for a k-torsor as well, thanks to the following result which is a straightforward generalization of [8, Lemme 6.7]).

Proposition 2.4. *Let G be a connected linear k-group and let X be a k-torsor under G. There is a canonical isomorphism* $\operatorname{UPic}(\overline{X}) \xrightarrow{\sim} \operatorname{UPic}(\overline{G})$, *functorial in G and X, in the derived category of discrete Galois modules.*

Combining the fact that $\text{Ш}^1_\omega(k, \operatorname{Pic}(\overline{X}_c)) = H^1(k, \operatorname{Pic}(\overline{X}_c))$ for any smooth compactification $\overline{X}_c$ of a k-torsor X under G (cf. [3]) with Proposition 1.3, Proposition 2.4, and Corollary 2.2, we obtain a new proof of the following result.

Corollary 2.5 *(Borovoi–Kunyavskiĭ [2]). With G and X as above,* $H^1(k, \operatorname{Pic}(\overline{X}_c)) \simeq \text{Ш}^2_\omega(k, \pi_1(\overline{G})^D)$.

3. Homogeneous spaces

Let G be a connected k-group such that $\operatorname{Pic}(\overline{G}) = 0$ (i.e. $(G^{\mathrm{red}})^{\mathrm{ss}}$ is simply connected). Let X be a homogeneous space of G defined over k. Let $\bar{x} \in X(\bar{k})$, and let $\overline{H}$ be the stabilizer of $\bar{x}$ in $\overline{G}$. Then $\operatorname{Gal}(\bar{k}/k)$ acts on $\mathbf{X}^*(\overline{H})$. We do not assume that X has a k-point or that $\overline{H}$ is connected.

Theorem 3.1. *For G and X as above, there is an isomorphism*

$$\operatorname{UPic}\big(\overline{X}\big) \xrightarrow{\sim} \big(\mathbf{X}^*\big(\overline{G}\big) \to \mathbf{X}^*\big(\overline{H}\big)\big) \quad (\textit{with } \mathbf{X}^*\big(\overline{G}\big) \textit{ in degree } 0)$$

in the derived category of discrete Galois modules. In particular, there is an exact sequence

$$0 \to U\big(\overline{X}\big) \to \mathbf{X}^*\big(\overline{G}\big) \to \mathbf{X}^*\big(\overline{H}\big) \to \operatorname{Pic}\big(\overline{X}\big) \to 0.$$

The exact sequence of Theorem 3.1 generalizes an exact sequence of Fossum–Iversen [6, Proposition 3.1] and Sansuc [8, Proposition 6.10]. Note that the requirement $\mathrm{Pic}(\overline{G})=0$ is not a serious restriction, since for any connected k-group G we can find a surjective homomorphism $G' \twoheadrightarrow G$ with $\mathrm{Pic}(\overline{G}')=0$.

Corollary 3.2. *For G and X as above there are injections* $\mathrm{Pic}(X) \hookrightarrow H^1(k, \mathbf{X}^*(\overline{G}) \to \mathbf{X}^*(\overline{H}))$ *and* $\mathrm{Br}_{\mathrm{a}}(X) \hookrightarrow H^2(k, \mathbf{X}^*(\overline{G}) \to \mathbf{X}^*(\overline{H}))$*, which are isomorphisms if $X(k) \neq \emptyset$.*

The corollary follows from Theorem 3.1 and Lemma 1.2.

4. The elementary obstruction

Let X be a k-variety. We have an extension of complexes of Galois modules

$$0 \to \bar{k}^\times \to \big(\bar{k}(\overline{X})^\times \to \mathrm{Div}(\overline{X})\big) \to \big(\bar{k}(\overline{X})^\times/\bar{k}^\times \to \mathrm{Div}(\overline{X})\big) \to 0.$$

It defines an element $e(X) \in \mathrm{Ext}^1(\mathrm{UPic}(\overline{X}), \bar{k}^\times)$. If X has a k-point, then this extension splits (in the derived category), hence $e(X)=0$. By slight abuse of terminology we call this class $e(X)$ the *elementary obstruction* to the existence of a k-point in X (cf. [5, Définition 2.2.1 and Proposition 2.2.4]).

When X is a k-torsor under a k-group G, Proposition 2.4 and Theorem 2.1 give us that $\mathrm{UPic}(\overline{X}) = \pi_1(\overline{G})^D$. We obtain

$$\mathrm{Ext}^1\big(\mathrm{UPic}(\overline{X}), \bar{k}^\times\big) = H^1\big(k, \mathrm{Hom}\big(\pi_1(\overline{G})^D, \bar{k}^\times\big)\big) = H^1\big(k, \mathbf{X}_*\big(T^{\mathrm{sc}}\big) \otimes \bar{k}^\times \to \mathbf{X}_*(T) \otimes \bar{k}^\times\big) = H^1\big(k, T^{\mathrm{sc}} \to T\big)$$

(where T^{sc} is in degree -1). The abelian group $H^1_{\mathrm{ab}}(k,G) := H^1(k, T^{\mathrm{sc}} \to T)$ is called the first abelian Galois cohomology group of G, and in [1] an abelianization map $\mathrm{ab}^1 : H^1(k,G) \to H^1_{\mathrm{ab}}(k,G)$ was constructed. Here we compute the elementary obstruction $e(X) \in H^1_{\mathrm{ab}}(k,G)$ in terms of the cohomology class $\mathrm{cl}(X) \in H^1(k,G)$.

Theorem 4.1. *Let X be a k-torsor under a connected k-group G. With the above notation we have $e(X) = \mathrm{ab}^1(\mathrm{cl}(X))$* (*up to sign*).

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