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Partial Differential Equations

CR-invariants and the scattering operator for complex manifolds with CR-boundary

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Abstract

Suppose that M is a CR manifold bounding a compact complex manifold X. The manifold X admits an approximate Kähler–Einstein metric g which makes the interior of X a complete Riemannian manifold. We identify certain residues of the scattering operator as CR-covariant differential operators and obtain the CR Q-curvature of M from the scattering operator as well. Our results are an analogue in CR-geometry of Graham and Zworski's result that certain residues of the scattering operator on a conformally compact manifold with a Poincaré–Einstein metric are natural, conformally covariant differential operators, and the Q-curvature of the conformal infinity can be recovered from the scattering operator. ***To cite this article: P.D. Hislop et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Des CR-invariants et la matrice de diffusion pour des variétés complexes avec CR-frontière. Soit M une variété CR qui est aussi la frontière d'une variété complexe et compacte X. Il y a une métrique g de type Kähler–Einstein sur X telle que $\mathrm{Int}(X)$ est une variété riemannienne complète. Nous étudions la matrice de diffusion sur (X, g) et nous montrons que les résidus à certains points sont des opérateurs différentiels CR-covariants. Nous montrons aussi qu'on peut recuperer la courbure CR Q en utilisant la matrice de diffusion. Nos résultats sont les analogues des résultats de Graham–Zworski pour le cas réel et asymptotiquement hyperbolique. ***Pour citer cet article : P.D. Hislop et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

In this Note we describe certain CR-invariants of a strictly pseudoconvex CR manifold M as residues of the scattering operator for the Laplacian on an ambient complex Kähler manifold X having M as a 'CR-infinity'. We also characterize the CR Q-curvature in terms of the scattering operator. Details will appear in [12]. Our results parallel earlier results of Graham and Zworski [10], who showed that if X is an asymptotically hyperbolic manifold carrying a Poincaré–Einstein metric, the Q curvature and certain conformally covariant differential operators on the 'conformal infinity' M of X can be recovered from residues of the scattering operator on X.

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To describe our results, we first recall some basic notions of CR geometry and recent results [4,6] concerning CR-covariant differential operators and CR-analogues of Q-curvature. If M is a smooth, orientable manifold of real dimension $(2n+1)$, a *CR-structure* on M is a real hyperplane subbundle H of TM together with a smooth bundle map $J : H \to H$ with $J^2 = -1$ that determines an almost complex structure on H. We denote by $T_{1,0}$ the eigenspace of J on $H \otimes \mathbb{C}$ with eigenvalue $+\mathrm{i}$; we will always assume that the CR-structure on M is integrable in the sense that $[T_{1,0}, T_{1,0}] = T_{1,0}$. We will assume that M is orientable, so that the line bundle $H^\perp \subset T^*M$ admits a nonvanishing global section. A *pseudohermitian structure* on M is smooth, nonvanishing section θ of $H^\perp$. The Levi form of θ is the Hermitian form $L_\theta(v, w) = \mathrm{d}\theta(v, Jw)$ on H. Strict pseudoconvexity of the CR structure on M means that the Levi form is positive definite. Thus, θ is a contact form, and the form $\omega = \theta \wedge (\mathrm{d}\theta)^n$ is a volume form that defines a natural inner product on $\mathcal{C}^\infty(M)$ by integration. The pseudohermitian structure on M also determines a connection on TM, the Tanaka–Webster connection ∇_θ; the basic data of pseudohermitian geometry are the curvature and torsion of this connection (see [17,19]).

Given a fixed CR-structure (H, J) on M, any nonvanishing section $\bar\theta$ of $H^\perp$ takes the form $\mathrm{e}^{2\Upsilon}\theta$ for a fixed section θ of $H^\perp$ and some function $\Upsilon \in \mathcal{C}^\infty(M)$. The corresponding Levi form is given by $\overline{L_\theta} = \mathrm{e}^{2\Upsilon} L_\theta$. In this sense the CR-structure determines a conformal class of pseudohermitian structures on M.

For strictly pseudoconvex domains, Fefferman and Hirachi [4] proved the existence of CR-covariant differential operators P_k of order $2k$, $k = 1, 2, \ldots, n+1$, whose principal parts are Δ_θ^k, where Δ_θ is the sub-Laplacian on M with respect to the pseudohermitian structure θ; Graham and Gover [6] proved that the same is true if M is a strictly pseudoconvex CR manifold. These authors exploit the Graham–Fefferman construction of conformally covariant differential operators [7] and Fefferman's construction of a circle bundle $\mathcal{C}$ over M with a natural conformal structure. Indeed, there is a mapping $\theta \mapsto g_\theta$ of a conformal class of pseudohermitian structures on M onto a conformal class of Lorentz metrics on $\mathcal{C}$. The operators P_k are pullbacks to M of the GJMS [8] conformally covariant differential operators on $\mathcal{C}$. The CR Q-curvature may be similarly defined as a pullback to M of Branson's Q-curvature [1] on the circle bundle $\mathcal{C}$. Here we will show that the operators P_k on M occur as residues for the scattering operator associated to a natural scattering problem with M as the boundary at infinity, and that the CR Q-curvature Q_θ^{CR} can be computed from the scattering operator as well.

To describe the scattering problem, we first discuss its geometric setting. Recall that if M is an integrable CR-manifold of real dimension $(2n+1)$ with $n \geqslant 2$, there is a complex manifold X of complex dimension $N = n+1$ having M as its boundary so that the CR-structure on M is that induced from the complex structure on X (this result is false, in general, when $n = 1$; see [11]). Let ρ be a defining function for M and denote by $\mathring{X}$ the interior of X (we take $\rho < 0$ in $\mathring{X}$). The associated Kähler metric g on X is the Kähler metric with Kähler form $-\frac{\mathrm{i}}{2}\partial\bar\partial \log(-\rho)$. The metric has the form $g = -\eta\rho^{-1} + (1 - r\rho)\rho^{-2}(\mathrm{d}\rho^2 + \Theta^2)$, where $\Theta|_M = \theta$, and $\eta|_H = h$ are the induced contact form on M and pseudohermitian metric on H, and r is a smooth function, the transverse curvature, which depends on the choice of ρ (see [9]). Thus, the conformal class of a pseudohermitian metric h on H, a *subbundle* of TM, is a kind of 'Dirichlet datum at infinity' for the metric g, that is $-\rho g|_H = h$.

It is natural to consider scattering theory for the Laplacian, Δ_g, on $(\mathring{X}, g)$. If we define $\rho = -x^2$, the metric g is seen to belong to the class of Θ-metrics considered by Epstein, Melrose, and Mendoza [2], so that the full power of their analysis of the resolvent $R(s) = (\Delta_g - s(N-s))^{-1}$ of Δ_g is available to study scattering theory on $(\mathring{X}, g)$. For $f \in \mathcal{C}^\infty(M)$, $\mathrm{Re}(s) = N/2$, and $s \neq N/2$, there is a unique solution u of $\Delta_g u = s(N-s)u$ with $u = (-\rho)^{N-s}F + (-\rho)^s G$, where $F, G \in \mathcal{C}^\infty(X)$, and $F|_M = f$. The uniqueness depends on absence of L^2 solutions of the eigenvalue problem for $\mathrm{Re}(s) = N/2$, which may be proved, for example, using [18]. The explicit formulas for the Kähler form and Laplacian obtained in [9] are used to obtain the asymptotic expansions of solutions to the generalized eigenvalue problem.

Unicity for the 'Dirichlet problem' defined above implies that the Poisson map $\mathcal{P}(s) : \mathcal{C}^\infty(M) \ni f \to u \in \mathcal{C}^\infty(\mathring{X})$ and the scattering operator $S_X(s) : \mathcal{C}^\infty(M) \ni f \to G|_M \in \mathcal{C}^\infty(M)$ are well-defined. The operator $S_X(s)$ depends a priori on the boundary defining function ρ for M. If $\bar\rho = \mathrm{e}^\varphi \rho$ is another defining function for M and $\varphi|_M = \Upsilon$, the corresponding scattering operator $\overline{S}_X(s)$ is given by $\overline{S}_X(s) = \mathrm{e}^{-s\Upsilon} S_X(s)\, \mathrm{e}^{(s-N)\Upsilon}$. The operator $S_X(s)$ admits a meromorphic continuation to the complex plane, possibly with essential singularities at the points $s = 0, -1, -2, \ldots$; see [15] where the scattering operator is described and the problem of studying its poles and residues is posed. The scattering operator is self-adjoint for s real.

Fefferman [3] constructed a local approximate solution to the complex Monge–Ampère equation near the boundary of a strictly pseudoconvex domain which is the Kähler potential of an approximate Kähler–Einstein metric; using the invariance properties of the Monge–Ampère operator, one can globalize this construction to obtain an approximate solution of the complex Monge–Ampère equation near the boundary of a compact complex manifold with boundary [5]. It follows that $\mathring{X}$ carries an approximate Kähler–Einstein metric g in the sense that $\mathrm{Ric}(g) = -(n+2)\omega + \mathcal{O}(\rho^{n+1})$, where Ric is the Ricci form. Our first result is:

Theorem 1. *Let X be a complex manifold of complex dimension $N = n+1$ with strictly pseudoconvex boundary M of real dimension $2n+1$. Let g be the Kähler metric on X described above, and let $S_X(s)$ be the scattering operator for Δ_g. Finally, suppose that Δ_X has no L^2-eigenvalues. Then $S_X(s)$ has poles at the points $s = (n+1)/2 + k/2$, $k \in \mathbb{N}$, whose residues are differential operators of order $2k$. If g is an approximate Kähler–Einstein metric, then for $1 \leqslant k \leqslant n+1$, these residues are the CR-covariant differential operators P_k up to a universal constant c_k.*

It follows from the self-adjointness (s real) and conformal covariance of $S_X(s)$ that the operators P_k are self-adjoint and conformally covariant. As in [10], the analysis centers on the Poisson map $\mathcal{P}(s)$ already defined. As shown in [2], the Poisson map is analytic in s for $\mathrm{Re}(s) > N/2$. Moreover, at the points $s = N/2 + k/2$, $k = 1, 2, \ldots,$ the Poisson operator takes the form $\mathcal{P}(s)f = (-\rho)^{N/2-k/2}F + [(-\rho)^{N/2+k/2}\log(-\rho)]G$, for functions $F, G \in \mathcal{C}^\infty(X)$ with $F|_M = f$, and $G|_M = c_k P_k f$. Here P_k are differential operators determined by a formal power series expansion of the Laplacian. An important ingredient in the analysis is the asymptotic form of the Laplacian due to Lee and Melrose [14] and refined by Graham and Lee in [9]. If the defining function ρ is an approximate solution of the complex Monge–Ampère equation, the differential operators P_k, $1 \leqslant k \leqslant N$, can be identified with the GJMS operators owing to the characterization of $\mathcal{P}(s)f$ described above (see Proposition 5.4 in [6]; the argument given there for pseudoconvex domains easily generalizes to the present setting).

Explicit computation shows that, for an approximate Kähler–Einstein metric g, the first operator has the form $P_1 = c_1(\Delta_b + n(2(n+1))^{-1}R)$, where Δ_b is the sub-Laplacian on X and R is the Webster scalar curvature, i.e., P_1 is the CR-Yamabe operator of Jerison and Lee [13]. The CR Q-curvature is a pseudohermitian invariant realized as the pullback to M of the Q-curvature of the circle bundle $\mathcal{C}$.

Theorem 2. *Suppose that X is a complex manifold with strictly pseudoconvex boundary M, equipped with an approximate Kähler–Einstein metric. Let $S_X(s)$ be the associated scattering operator. The formula $c_N Q_\theta^{\mathrm{CR}} = S_X(N)1$ holds.*

It follows from Theorem 1 and the conformal covariance of $S_X(s)$ that if $\bar{\theta} = \mathrm{e}^{2\Upsilon}\theta$, then $\mathrm{e}^{2N\Upsilon} Q_{\bar{\theta}}^{\mathrm{CR}} = Q_\theta^{\mathrm{CR}} + P_N \Upsilon$ as was already shown in [4]. From this it follows that the integral $\int_M Q_\theta^{\mathrm{CR}}\,\mathrm{d}\omega$ is a CR-invariant. We remark that the integral of Q_θ^{CR} vanishes for any three-dimensional CR-manifold because the integrand is a total divergence (see [4], Proposition 3.2 and comments below), while for any hypersurface in $\mathbb{C}^N$, there is a pseudohermitian structure for which $Q_\theta^{\mathrm{CR}} = 0$ (see [4], Proposition 3.1). Thus it is not clear at present under what circumstances this invariant is nontrivial.

If ρ is the defining function associated to an approximate Kähler–Einstein metric on X, the volume of the set $\{-\rho < \varepsilon\}$ has an asymptotic expansion of the form $c_0\varepsilon^{-n-1} + c_1\varepsilon^{-n} + \cdots + c_n\varepsilon^{-1} + L\log(-\varepsilon) + V + \mathrm{o}(1)$. Seshadri [16] showed that L is, up to a constant, the integral of Q_θ^{CR}; in [12] we give an independent proof using scattering theory along the lines of [10].

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