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Numerical Analysis

Improved interface conditions for a non-overlapping domain decomposition of a non-convex polygonal domain

Chokri Chniti [a], Frédéric Nataf [a], Francis Nier [b]

[a] *CMAP, École polytechnique, 91128 Palaiseau cedex, France*
[b] *IRMAR, campus de Beaulieu, 35042 Rennes cedex, France*

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Abstract

We propose a local improvement of domain decomposition methods which fits with the singularities occurring in the solutions of elliptic equations in polygonal domains. This short presentation focuses on a model elliptic problem with the decomposition of a non-convex polygonal domain into convex polygonal subdomains. After explaining the strategy and the theoretical design of adapted interface conditions at the corner, we present numerical experiments which show that these new interface conditions satisfy some optimality properties. ***To cite this article: C. Chniti et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Conditions d'interface améliorées pour une décomposition d'un domaine polygonal non convexe. Dans cette Note nous proposons une amélioration locale des méthodes de décomposition de domaine adaptée aux singularités présentes dans les solutions de problèmes elliptiques dans des domaines polygonaux. Cette courte présentation se limite à un problème elliptique modèle où un domaine polygonal non convexe est décomposé en sous-domaines convexes. Après avoir brièvement présenté la stratégie et la détermination théorique des conditions d'interface adaptées au coin, nous présentons des résultats numériques qui montrent que ces nouvelles conditions d'interface vérifient des propriétés d'optimalité. ***Pour citer cet article : C. Chniti et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

Domain decomposition methods are now well understood in the case of a regular domain decomposed into regular subdomains, see for example [5]. A significant challenge for the applications is a good understanding of the singular cases: problems with corners in 2D. The general principle of those methods is as follows: for an elliptic operator L, a domain Ω and a given right-hand side f, consider the problem of finding u such that

$$Lu = f \quad \text{in } \Omega \quad + \quad \text{B.C.} \quad \text{on } \partial\Omega. \tag{1}$$

E-mail addresses: chniti@cmap.polytechnique.fr (C. Chniti), nataf@cmap.polytechnique.fr (F. Nataf), Francis.Nier@univ-rennes1.fr (F. Nier).

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When the domain Ω is 'large', it can be decomposed into subdomains, $\overline{\Omega} = \bigcup_{i=1}^{i=N} \overline{\Omega_i}$ where Ω_i is an open subdomain of Ω. The initial problem (1) is then approximated by an iterative process:

$$\begin{cases} Lu_i^{n+1} = f & \text{in } \Omega_i, \\ B_{ij}\gamma_{ij}u_i^{n+1} = B_{ij}\gamma_{ij}u_j^n & \text{on } \partial\Omega_i \cap \overline{\Omega_j}\ (i \neq j), \\ +\text{B.C.} & \text{on } \partial\Omega_i \cap \partial\Omega \end{cases} \tag{2}$$

simultaneously for all $i = 1, \ldots, N$. The interface operators B_{ij} can be differential or pseudodifferential operators. The choice of the interface operators has a very great influence on the speed of convergence of the algorithm. Within the framework of the regular interfaces, a good final choice actually relies on a compromise between the theoretical optimality and the ease of implementation, see [2].

In a domain with a conical singularity (corner), it is known after Kondratiev [3] that even when the right-hand side of (1) vanishes at infinite order at the corner the solution may have singularities or more generally a non-trivial asymptotic expansion. Moreover, the first term of this asymptotic expansion at the corner depends strongly on the operator L, the geometry which is reduced to the corner angle for first approximations, and the boundary conditions. A priori the singularities in the subdomains Ω_i, $i = 1, \ldots, N$, do not coincide with the ones of the whole domain Ω. The diagnosis of a locally slower convergence of domain decomposition algorithm in the presence of conical singularities in [4] invoked this bad matching. The present work again makes use of the flexibility in the choice of the interface boundary conditions B_{ij}, on which the singularities depend, in order to improve the convergence in such cases.

2. Interface conditions of order 2

Our approach consists in keeping the good interface conditions for smooth boundaries far from the corner with an adaptation in the vicinity of the corner (see [1] for details). For a second order elliptic differential operator L, those good second order interface conditions B_{ij} have the form $\frac{\partial}{\partial n} + \beta - \frac{\partial}{\partial \tau}(\frac{\alpha}{2}\frac{\partial}{\partial \tau})$, with constants $\alpha, \beta \in \mathbb{R}_+$ and where τ denotes the tangential variable. For $\Omega = \mathbb{R}^2$ split into two half-planes, the choice of α and β is optimized according to the mesh size h in the tangential direction which brings a natural upper bound for numerical frequencies. By introducing $e_i^n(x, \tau) = u_i^n - u$ and its Fourier transform in the tangential variable $\hat{e_i}(x, k)$, the optimized coefficients $\alpha_{\text{opt}}, \beta_{\text{opt}} > 0$ are determined in [2] by the max-min principle

$$\min_{\alpha,\beta\in\mathbb{R}} \max_{|k|\leqslant\pi/h} \left|\rho(k; \alpha, \beta)\right| \tag{3}$$

with $\rho(k; \alpha, \beta) = \frac{\hat{e_i}^{n+2}(0,k)}{\hat{e_i}^{n}(0,k)}$. At a corner and by taking polar coordinates the boundary operator takes the form $B_{ij} = \frac{\partial}{r\partial\theta} + \beta_{\text{opt}} - \frac{\alpha_{\text{opt}}}{2}\frac{\partial^2}{\partial r^2}$. In the asymptotic $r \to 0$, $\frac{\partial}{r\partial\theta}$ and $\frac{\partial}{\partial r}$ are operators of order -1 and the principal part of B_{ij} is reduced to the last term $-\frac{\alpha_{\text{opt}}}{2}\frac{\partial^2}{\partial r^2}$. The absence of a normal derivative in this principal part means that asymptotically the interface conditions behave like Dirichlet interface conditions and do not transmit well the information from one subdomain to its neighboring ones. One way to solve this problem is by using at the corner an interface condition like $\frac{\partial}{r\partial\theta} + \frac{\beta_1}{r} - \frac{\partial}{\partial r}\frac{\alpha_1 r}{2}\frac{\partial}{\partial r}$ where all the terms have the homogeneity order -1, see [4]. A synthesis is done by taking $B_{ij} = \frac{\partial}{r\partial\theta} + \tilde{\beta}(r) - \frac{\partial}{\partial r}\frac{\tilde{\alpha}(r)}{2}\frac{\partial}{\partial r}$ where

$$\tilde{\alpha}(r) = \begin{cases} \alpha_1 r & \text{if } r \leqslant \dfrac{\alpha_{\text{opt}}}{\alpha_1}, \\ \alpha_{\text{opt}} & \text{if } r \geqslant \dfrac{\alpha_{\text{opt}}}{\alpha_1}, \end{cases} \quad \text{and} \quad \tilde{\beta}(r) = \begin{cases} \frac{\beta_1}{r} & \text{if } r \leqslant \dfrac{\beta_1}{\beta_{\text{opt}}}, \\ \beta_{\text{opt}} & \text{if } r \geqslant \dfrac{\beta_1}{\beta_{\text{opt}}}, \end{cases} \tag{4}$$

with $\alpha_1, \beta_1 > 0$. In a numerical implementation, the coefficients α_1, β_1 are such that the matching radius $r_0 = \min\{\frac{\alpha_{\text{opt}}}{\alpha_1}, \frac{\beta_1}{\beta_{\text{opt}}}\}$ corresponds to three meshes of the discretized domain. This provides the first relation between α_1 and β_1.

3. Theoretical determination of a good pair (α_1, β_1) in a subdomain for a model problem

The domain Ω is the sector $\Omega = \{(r\cos\theta, r\sin\theta), r > 0, \theta_0 < \theta < \theta_+\}$, with $\theta_+ - \theta_0 \in (0, 2\pi)$. Consider the homogeneous Dirichlet problem

$$(\eta - \Delta)u = f, \quad u|_{\theta=\theta_0} \equiv 0, \quad u|_{\theta=\theta_+} \equiv 0, \tag{5}$$

which is well posed in $H^1(\Omega)$ for $f \in L^2(\Omega)$. For a trial function v, set $e = u - v$. Then Kondratiev theory [3] says that even when the consistency is verified at infinite order at $r = 0$, $(\eta - \Delta)e = \mathrm{O}(r^\infty)$, the conclusion is that $\exists a_0$ s.t.

$$e(r,\theta) = a_0 r^{1/x_0} \sin\left(\frac{\theta - \theta_0}{x_0}\right) + \mathrm{o}\left(r^{1/x_0}\right) \tag{6}$$

with $x_0 = \frac{\theta_+ - \theta_0}{\pi}$. The function $r^{1/x_0}\sin(\frac{\theta-\theta_0}{x_0})$ is the main natural singularity attached to the boundary value problem (5). It also provides the first term in the asymptotic expansion of the solution u to (5) with a vanishing right-hand side f.

The sector $\Omega = \mathbb{R}_+^* \times (\theta_0, \theta_+)$ in polar coordinates is decomposed into $\Omega_1 = \mathbb{R}_+^* \times (\theta_-, \theta_+)$ and $\Omega_2 = \mathbb{R}_+^* \times (\theta_0, \theta_-)$, with $\theta_0 < \theta_- < \theta_+$. We focus on the subdomain Ω_1, the treatment of Ω_2 is similar. The boundary problem (2) in Ω_1 with the interface conditions (4) solved by the error $e_1^{n+1} = u_1^{n+1} - u$ reads

$$\begin{cases} \left(\eta - \dfrac{1}{r^2}\left((r\partial_r)^2 + \partial_\theta^2\right)\right) e_1^{n+1}(r,\theta) = 0, \\ e_1^{n+1}(r,\theta_+) = 0, \\ \left(-\dfrac{1}{r}\partial_\theta + \tilde\beta(r) - \dfrac{1}{2}\partial_r\left(\tilde\alpha(r)\partial_r\right)\right) e_1^{n+1}(r,\theta_-) = \left(-\dfrac{1}{r}\partial_\theta + \tilde\beta(r) - \dfrac{1}{2}\partial_r\left(\tilde\alpha(r)\partial_r\right)\right) e_2^{n}(r,\theta_-). \end{cases} \tag{7}$$

This problem admits a well posed variational formulation in a subspace of $H^1(\Omega_1)$ with the sign conditions α_1, $\beta_1 > 0$. The main singularities associated with this problem are derived following [3] by considering the principal part as $r \to 0$ and by applying the Mellin transform, $\hat u(z) = \int_0^\infty r^{\mathrm{i}z} u(r) \frac{\mathrm{d}r}{r}$.

We are led to consider the system

$$\left(\partial_\theta^2 - z^2\right)\hat e(z,\theta) = 0, \quad \hat e(z,\theta_+) = 0, \quad \left(\partial_\theta - \beta_1 - \frac{\alpha_1}{2}z^2\right)\hat e(z,\theta_-) = \hat g(z), \tag{8}$$

whose solution is $a(z)\,\mathrm{e}^{z(\theta-\theta_-)} + b(z)\,\mathrm{e}^{-z(\theta-\theta_+)}$ with

$$a(z) = \mathcal{R}(z)\hat g(z), \quad b(z) = -a(z)\,\mathrm{e}^{z(\theta_+-\theta_-)},$$
$$\mathcal{R}(z) = \left[\left(z - \beta_1 - \frac{\alpha_1}{2}z^2\right) + \left(z + \beta_1 + \frac{\alpha_1}{2}z^2\right)\mathrm{e}^{2z(\theta_+-\theta_-)}\right]^{-1}.$$

Proposition 3.1. *The poles with a positive imaginary part of the factor $\mathcal{R}(z)$ are the purely imaginary complex numbers $z = \mathrm{i}t$, with $t > 0$ and $\tan(t(\theta_+ - \theta_-)) = \frac{2t}{\alpha_1 t^2 - 2\beta_1}$ whose first positive solution is denoted by t_1.*

Hence the main singularity which can be generated by solving (7) is $\mathrm{O}(r^{t_1})$. It is an artificial singularity depending on the domain decomposition. Here comes the strategy of the convergence improvement: the algorithm must not produce artificial singularities, in particular when the solutions in subdomains have the right asymptotic behaviour. In our case it is namely when the function e_2^n defined in Ω_2 has the form (6). By forgetting the $\mathrm{o}(r^{\frac{1}{x_0}})$ remainder, and by taking for $\hat g$ the Mellin transform of the right-hand side of (7), this provides the condition $\hat g(\mathrm{i}t_1) = 0$. A simple calculation gives

$$-\beta_1 + \frac{\alpha_1}{2x_0^2} = \frac{1}{x_0\tan(\pi x/x_0)}, \quad x = \frac{\theta_+ - \theta_-}{\pi}. \tag{9}$$

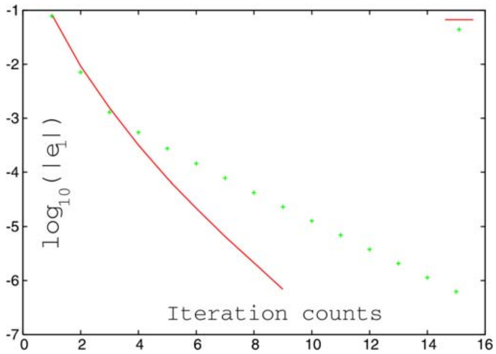

Fig. 1. **COC** (solid line), **CICC** (dotted line).

Table 1
Number of iterations for different value of β_1/α_1, with $|error|_1 < 10^{-6}$

β_1/α_1	0.05	0.1	2/9	1	2	5	10
Iteration count	16	13	9	12	24	59	116

4. Numerical experiments

The previous strategy was tested on various examples. We show the simple case for a L-shaped domain where $\theta_0 = -3\pi/2$, $\theta_+ = 0$ and $\theta_- \in (-\pi, -\pi/2)$. In a symmetric decomposition where $\theta_- = -3\pi/4$, the relation (9) says $\beta_1 = \frac{2\alpha_1}{9}$ for both subdomains. Using `freeFEM++`, we present a numerical comparison between the optimized conditions of the regular case and the new conditions in the vicinity of the corner. We note $|u|_1 = (\int_\Omega |\nabla u|^2(x)\,\mathrm{d}x)^{1/2}$. We denote **COC** the new interface condition with coefficients given by (9), and **CICC** the interface conditions with constant coefficients up to the corner. If we use $\frac{\beta_1}{\alpha_1} = \frac{2}{9}$, the iteration count is equal to 9 and it's optimal, see Table 1. With **CICC**, 15 iterations are necessary instead of 9 with **COC**, see Fig. 1. These results show that this choice is actually numerically optimal.

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