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Partial Differential Equations

Application of the exact null controllability of the heat equation to moving sets

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Abstract

We study the lagrangian controllability of the heat equation in several dimensions. In dimension one, we prove that any pairs of intervals are diffeomorphic through the flow of the solution of the heat equation via an adequate control. In higher dimensions we prove a similar controllability result for the flow of the gradient of the solution in a radial case in arbitrary finite time, and for convex domains in a sufficiently large time. ***To cite this article: T. Horsin Molinaro, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Application de la contrôlabilité exacte à zéro de l'équation de la chaleur au déplacement d'ensembles. On étudie la contrôlabilité lagrangienne de l'équation de la chaleur en toutes dimensions. En dimension 1, on montre que deux intervalles quelconques sont difféomorphes via le flot de la solution de l'équation de la chaleur avec un contrôle adéquat. En dimension supérieure on prouve un résultat de contrôlabilité similaire pour le flot du gradient, en temps fini fixé pour le cas radial, et en temps assez grand pour le cas convexe. ***Pour citer cet article : T. Horsin Molinaro, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

For many years, the controllability of evolution equations modeling (usually) physical systems has been studied; see, e.g. [6,9,10,13,15]. Concerning the more specific case of fluids, ideas have been developed to control the speed of the fluid (see, e.g. [3] and [4] for global results) and, more recently, in fluid-interaction problems, the position of a body in a fluid and the velocity of the fluid itself (see, e.g. [1,14]). Considering applications, for instance, in the study of the dispersion of particles, an important tool to approach pollution problems, it might be useful to control the position of the fluid itself, and not only its velocity. This work is concerned with the problem of moving subsets inside a domain under the flow of the solution (or the gradient of the solution) of a distributed controlled heat equation. In Section 2 we give the result in the one-dimensional case, i.e. Theorem 1 which is explained in Section 3. In the final Section 4 we consider the case of higher dimensions where we prove an similar result for the flow of the gradient.

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2. Formulation of the problem, statement of the result

Let $I = (0, 1)$ and let I_1 and I_2 two given closed intervals of I such that $\partial I_1 \cap \partial I = \partial I_2 \cap \partial I$. Let also ω be an open set such that $\overline{\omega} \subset I$. For a given $T > 0$ we want to find h (whose regularity will specified later) such that the solution u of

$$u_t - u_{xx} = h\mathbb{1}_{(0,T)\times\omega} \quad \text{in } (0, T) \times I, \qquad u = 0 \quad \text{on } (0, T) \times \partial I, \qquad u(t = 0) = 0, \tag{1}$$

moves I_1 to I_2 from $t = 0$ to $t = T$.

Definition. We will say that u 'moves' I_1 to I_2 from $t = 0$ to $t = T$, if once considered $g : [0, T] \times [0, 1] \to [0, 1]$ the solution of

$$\partial_t g(t, x) = u\big(t, g(t, x)\big), \quad (t, x) \in [0, T] \times [0, 1], \qquad g(0, x) = x, \tag{2}$$

one has

$$g(T, I_1) = I_2, \tag{3}$$

and $g(T, \cdot)$ is a diffeomorphism of I_1 onto I_2.

We prove the following:

Theorem 1. *There exists $h \in L^2((0, T) \times \omega)$ such that if u is the solution of* (1) *then u moves I_1 to I_2 from $t = 0$ to $t = T$. Moreover, if one has T large enough* (*depending on $d(I_2, \mathbb{R} \setminus I)$*)*, one can also impose $u(T, \cdot) = 0$.*

3. Sketch of proof of Theorem 1

Without loss of generality we may assume that $\omega = (a, b)$. We will assume that $\min(\min(I_1), \min(I_2)) > b$. If it is not, minor modifications are needed. The rough strategy is the following: we will move first I_1 into a subinterval of ω with a solution of the heat equation for $t \in [0, T/3]$. We then perform a similar procedure to move backwards I_2 into a subinterval of ω from $t = T$ to $t = 2T/3$. Then we will use the exact controllability of the heat equation to combine the two first arguments between $t = T/3$ and $t = 2T/3$.

For the purpose of the first step we consider u_1 the solution of (1) with $h = -1$. Let us also consider for a small $\mu \in (0, \min(T/6, (b - a)/4))$ a nonnegative function $\eta(t, x)$ such that η is decreasing with respect to t for fixed x, and such that

$$\begin{cases} \eta(t, x) = 0, \quad t \leqslant T/3 - \mu,\ x \leqslant a + \mu/2; \quad \eta(t, x) = 1, \quad t \leqslant T/3 - \mu,\ x \geqslant a + \mu, \\ \eta(T/3, x) = 0, \quad x \leqslant b - \mu; \quad \eta(T/3, x) = 1, \quad x \geqslant b - \mu/2. \end{cases}$$

Let M be a positive constant. Then for M large enough $M\eta u_{-1}$ will move I_1 to $[a_2, b_2] \subset (a + \mu/2, b - \mu)$.

For the second step from $t = T$ to $t = 2T/3$ one applies a similar procedure by taking $u(x, t) = C(1 - x)$ with $C > 0$ large enough for $x \geqslant b$ and $u = 0$ for $x \leqslant a$ and suitable in ω so that the flow moves backwards I_2 to some $[a_3, b_3] \subset (a + \mu/2, b - \mu)$ and $u(2T/3, \cdot)$ also being 0 on $(0, b - \mu)$. Let us mention that proceeding here, as in the first step, is not possible due to the irreversibility of the heat equation.

Between $t = T/3$ to $t = 2T/3$ one connects (a, b) to itself with C^2 noncrossing curves such that these curves join $[a_2, b_2]$ to $[a_3, b_3]$ and are straight lines on $(a, a + \mu/2)$ and $(b - \mu, b)$. The speed of such a deformation defines u in $[T/3, 2T/3] \times (a, b)$ with support in $[a + \mu/2, b - \mu]$.

Now, due to the exact zero controllability (see [8,9,11]) and since $\Delta u(2T/3, \cdot)$ is zero except in $(b - \mu, b)$, there exists a control $h \in L^2(T/3, 2T/3; L^2(\omega'))$ with $\omega' = (b - \mu/2, b - \mu/4)$ such that one can drive $u(T/3, \cdot)$ to $u(2T/3, \cdot)$ with a solution of the corresponding controlled heat equation on $(T/3, 2T/3) \times \Omega'$ with Dirichlet boundary conditions where $\Omega' := (b - \mu, 1)$.

Thus we have a solution of (1) on $(0, T) \times \Omega$ which is continuous in time and space. Nevertheless, in order to obtain a control in L^2 in space and time it suffices to multiply u near $\bar{x} := b - \mu$ for $t \in (T/3, 2T/3)$ by a nonnegative function which is zero in a small neighborhood of $\bar{x}$ and which goes to 1 in a larger interval.

We skip the detailed proof of going to rest, but let us just explain the main idea: let $v(x) = x$ for $x \leqslant a$ and $v(x) = 1 - x$ for $x \geqslant b$. Then for any $\tau > 0$, v can be driven to 0 from $t = 0$ to $t = \tau$ with $h_\tau \in L^2((0, \tau); L^2(\omega))$.

The backward flow of I_2 is, due to the regularity, a subinterval I_2' of I. Now we perform a similar method as before on $[0, T]$, with $u(x,t) = v(x)$ for $t \in [2T/3, T]$, and if T is large enough I_2' will be steered into (a,b) from $t = T$ to $t = 2T/3$.

Remarks. When $\sup(I_1) < 1$ and $\sup(I_2) < 1$ with similar arguments, one can impose that $g(T,\cdot)$ is any given preserving order diffeomorphism of I_1 onto I_2, but we are note able to do it when $\sup(I_1) = \sup(I_2) = 1$. Let us also point out that the above method can be related to the so-called 'return method' introduced by J.-M. Coron in [2], since we go from rest to rest, meanwhile we get suitable properties during the motion.

Remark *(Extension to the nonlinear case).* Let us remark that this method can be reformulated in the framework of [5] for $u_t - u_{xx} = u|u|^{p-1} + h\mathbb{1}_{(0,T)\times\omega}$, for large enough T.

The case of initial moving states. It is straightforward that the same method applies if one considers $u(t=0) = u_0 \in L^2(0,1)$, but we think that for fluid models $u(t=0) = 0$ is more relevant.

4. Higher dimensions

Now let us illustrate the higher dimensions by considering the case where $\Omega = B(0,1)$, the open unit ball in $\mathbb{R}^N$, $N \geqslant 2$, and $\omega = B(0, 1/2)$ we take F_1 and F_2 two closed subset of $\Omega \setminus \omega$, and we want to study if there exists $h \in L^2(\omega \times (0,T))$ such that the solution of

$$\partial_t u - \Delta u = h\mathbb{1}_{\omega\times(0,T)}, \qquad u = 0 \quad \text{on } (0,T)\times\partial\Omega, \qquad u(t=0) = 0, \tag{4}$$

moves F_1 to F_2 through the flow of minus its gradient during an interval of time of length T. We will assume that F_1 and F_2 are the closure of two C^3 open sets that are C^3 isotopic up to the boundary. We prove the following result:

Theorem 2. *For any $T > 0$ there exists $h \in L^2((0,T)\times\omega)$ such that for the solution of* (4)*, the flow of $-\nabla u$ is well-defined for $x \in F_1$ and*

(i) *the image of x through this flow at time t is in F_2,* (ii) *and any point in F_2 is the image of such a x.*

Moreover if T is large enough then one can replace (ii) *by the condition $u(t=T) = 0$.*

The idea of the proof in the symmetric case is based on a similar procedure as the proof of Theorem 1. For $t \in [0, T/3]$ we move F_1 into $F_1' \subset \omega$ through minus the gradient of a radial solution (which can be shown to be radially convex away of ω) of the controlled heat equation that we adjust in monotone way inside ω to a constant so that the moved set goes to rest.

We proceed the same way between $(2T/3, T)$ with a modification in ω of the Green's function at the origin of the Laplace's equation in Ω. Outside of ω we apply the exact controllability to trajectories as in Section 3. In ω, between $t = T/3$ and $t = 2T/3$ we move F_1' into the backward image F_2' of F_2 at $t = 2T/3$ by the flow of a gradient $\nabla v(t,x)$. Let us just spot on the fact that to perform this, it suffices to move the boundary of F_1' to the boundary of F_2': let $\phi(t,\cdot) : \overline{\omega} \to \overline{\omega}$ be a C^3 families of diffeomorphism such that $\phi(2T/3, F_1') = F_2'$. One then construct $v(t,\cdot)$ such that for $\nabla v(t, \phi(t,x)) \cdot n(\phi(t,x)) = \partial_t\phi(t,x)\cdot n(\phi(t,x))$, for $x \in \partial F_t'$ (for example one can extend v regularly to $\overline{\omega}$ with $-\Delta v + v = 0$ in F_t' and $\partial_n v(t,y) = \partial_t\phi(t, \phi^{-1}(t,y))\cdot n(y)$ for any $y \in \partial F_t'$) where n is the outward normal unitary vector on $\partial F_t'$.

This concludes the first part of Theorem 2.

Obtaining $u(t=T) = 0$ as in Theorem 1 is possible but defining the flow of ∇u is only possible through the theory of renormalized solution (see [7]) for which we lose the topological properties of F_2 and thus we thus obtain the possibility of moving F_1 into F_2.

The radial symmetry is not essential. For the case of a convex Ω, instead of using Green's functions we use capacitory functions of balls. Let B be a ball compactly included in ω, its capacitory function c_B is known to have convex level sets and no singular points outside ω (see [12]) thus the flow of its (or minus its) gradient will drive any subset of $\Omega \setminus \omega$ into $\omega \setminus \overline{B}$ for a sufficiently long period $T/3$. However, since we start from rest at $t = 0$ we have to go close enough in C^1 norm to c_B in an interval $[0, T/3]$ with T large enough with a solution of the controlled heat equation. This is possible due to the asymptotic behavior of the heat equation, and during that period $T/3$, F_1 will have been moved to a subset of Ω. The remaining argument being similar as before, we have then:

Theorem 3. *When Ω is convex and ω is any open subset compactly included in Ω the result of Theorem 2 is true provided T is large enough.*

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