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Partial Differential Equations

Is it possible to cancel singularities in a domain with corners and cracks?

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Abstract

In a domain with corners, we prove that by acting on an arbitrarily small part of the domain or on a small part of the boundary, we obtain a regular solution of the Laplace equation. ***To cite this article: M.T. Niane et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Est-il possible de supprimer des singularités dans un domaine fissuré ? On montre que, dans un domaine à coins, par une action sur une petite partie du domaine ou sur une petite partie de la frontière, on obtient une solution régulière de l'équation de Laplace. ***Pour citer cet article : M.T. Niane et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

1. Introduction

Consider Laplace equation with Dirichlet boundary conditions in a domain $\Omega \subset \mathbb{R}^2$ with corners. Nonconvex angles of the boundary of Ω produce singularities even if the right-hand side of the equation is smooth (see [1] and [3]). Singularities are rarely desired (like in lightning conductors). So far, there is no way of killing singularities by acting on an arbitrarily small part of the domain. Here we propose a method to do so. The proof is based on a density result, on a bi-orthogonality property of the dual singular solutions and the unicity theorem of Holmgren and Cauchy–Kowalevska (see [2]).

Let $m+1$ be the number of nonconvex angles of the boundary of Ω. Let ϖ be a nonempty domain of Ω (see Fig. 1). We prove that there exist $m+1$ regular functions $(g_i)_{0\leqslant i\leqslant m}$ with compact support in ϖ such that for any $f \in L^2(\Omega)$, if $(c_i)_{0\leqslant i\leqslant m}$ are the singularity coefficients of problem:

$$\text{Find } v \in H_0^1(\Omega) \text{ such that } \quad -\Delta v = f \quad \text{in } \Omega, \tag{1}$$

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then, problem

$$\text{Find } y \in H_0^1(\Omega) \text{ such that} \quad -\Delta y = f - \sum_{i=0}^{m} c_i g_i \quad \text{in } \Omega, \tag{2}$$

has a unique solution y in $H^2(\Omega)$.

We also prove that if Γ_0 is an arbitrarily small open subset of the boundary Γ of Ω , there exist $m+1$ regular functions $(h_i)_{0\leqslant i\leqslant m}$ defined on Γ with compact support in Γ_0 such that problem

$$\begin{cases} \text{Find } y \in H^1(\Omega) \text{ such that} \\ -\Delta y = f \quad \text{in } \Omega, \quad y = \sum_{i=0}^{m} c_i h_i \quad \text{on } \Gamma, \end{cases} \tag{3}$$

has a unique solution y in $H^2(\Omega)$.

2. Density theorem

Let H be a Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle_H$.

Theorem 2.1 *(Density property). Let H be a Hilbert space, D a dense subspace of H and $\{e_0, \dots, e_m\}$ a linearly independent subset of H. Then, there exist $\{d_0, \dots, d_m\}$ in D such that $\forall i, j \in \{0, \dots, m\}$, $(e_i, d_j)_H = \delta_{ij}$.*

Proof. By Schmidt's orthogonalization, there exist $v_0, v_1, \dots, v_m$ such that $(v_i, e_j)_H = \delta_{ij}, \forall i, j = 0, \dots, m$. As D is dense in H, there exist sequences $(v_i^{(n)})$ of elements in D such that $v_i^{(n)} \to v_i$ in H as $n \to \infty$, for all $i = 0, \dots, m$. This implies that $(v_i^{(n)}, e_j)_H \to (v_i, e_j)_H = \delta_{ij}$ as $n \to \infty$, and for n large enough, the matrix $B_n = ((v_i^{(n)}, e_j)_H)_{0\leqslant i,j\leqslant m}$ is invertible. Fix such a n. Write $B_n^{-1} = (c_{ij})_{0\leqslant i,j\leqslant m}$. The requested elements are $d_i = \sum_{k=0}^{m} c_{ik} v_k^{(n)}$, since $(d_i, e_j)_H = \sum_{k=0}^{m} c_{ik} (v_k^{(n)}, e_j)_H = \delta_{ij}$. □

3. Bi-orthogonality property of harmonic functions

Theorem 3.1. *Let Ω be a nonempty domain of $\mathbb{R}^n$, ϖ a nonempty open subset of Ω. Assume that $\{w_0, \dots, w_m\}$ is a set of linearly independent harmonic functions of $L^2(\Omega)$. Then, there exist $\mathcal{C}^\infty$ functions $(g_i)_{0\leqslant i\leqslant m}$ with compact support in ϖ such that: $\forall i, j \in \{0, \dots, m\}$, $\int_\Omega w_i g_j \,\mathrm{d}x = \delta_{ij}$.*

Proof. Let $H = L^2(\varpi)$. Let us prove that $w_0|_\varpi, \dots, w_m|_\varpi$ are linearly independent. Assume that there exist real numbers $\alpha_0, \dots, \alpha_m$ such that: $\sum_{i=0}^{m} \alpha_i w_i = 0$ in ϖ. Since this latter sum is harmonic in Ω then $\sum_{i=0}^{m} \alpha_i w_i = 0$ in Ω. Therefore, $\alpha_0 = \dots = \alpha_m = 0$.

Since $\mathcal{D}(\varpi)$ is dense in $L^2(\varpi)$, then by Theorem 2.1, there exist $g_0, \dots, g_m \in \mathcal{D}(\Omega)$ with compact support in ϖ such that $\forall i, j \in \{0, \dots, m\}$, $\int_\Omega w_i\, g_j \,\mathrm{d}x = \delta_{ij}$. □

In the sequel, denote by ν the outer unit normal vector to Γ.

Theorem 3.2. *Let Ω be a nonempty domain of $\mathbb{R}^n$, Γ_0 be nonempty open and analytic subset of the boundary Γ of Ω. Suppose that $\{w_0, \dots, w_m\}$ is a set of linearly independent harmonic functions of $L^2(\Omega)$ such that:*

$$\forall i \in \{0, \dots, m\}, \quad w_i|_{\Gamma_0} = 0 \quad \textit{on } \Gamma_0, \quad \left.\frac{\partial w_i}{\partial \nu}\right|_{\Gamma_0} \in L^2(\Gamma_0).$$

Then there exist $\mathcal{C}^\infty$ functions $(h_i)_{0\leqslant i\leqslant m}$ with compact supports in Γ_0 such that:

$$\forall i, j \in \{0, \dots, m\}, \quad \int_\Gamma \frac{\partial w_i}{\partial \nu} h_j \,\mathrm{d}\sigma = \delta_{ij}.$$

Proof. The proof is based on the same principle as Theorem 3.1. □

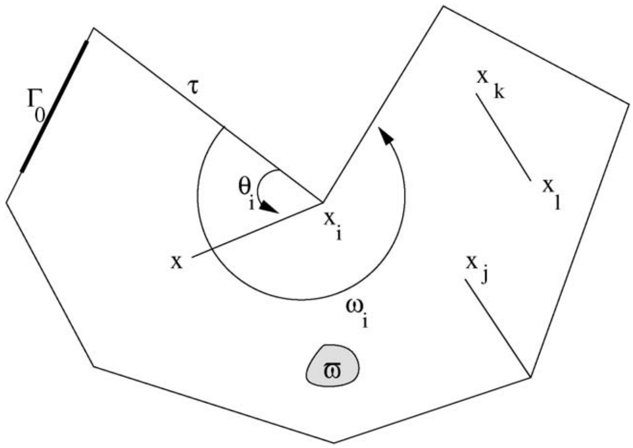

Fig. 1. Domain with corners and cracks.

4. Cancellation of singularities

4.1. Preliminary results on dual singular solutions

Denote by $\|.\|$ the Euclidean norm on $\mathbb{R}^2$. Consider a nonempty polygonal domain Ω of $\mathbb{R}^2$. Let $(x_i)_{0\leqslant i\leqslant m}$ be vertices of nonconvex angles $(\omega_i)_{0\leqslant i\leqslant m}$, say ω_i is greater than π. Let $(\theta_i)_{0\leqslant i\leqslant m}$ be the angle defined by vector $x - x_i$ and τ (see Fig. 1). Let $i \in \{0,\ldots,m\}$, denote by η_i a truncation function in a neighbourhood of the vertex x_i, whose support does not meet any other vertex than x_i, any other face than those whose intersection is x_i, and the support of Γ_0. Let w_i^* be the dual singular solution associated to the corner x_i. Thanks to Grisvard [1], we have $w_i^* = \|x - x_i\|^{-\frac{\pi}{\omega_i}} \sin(\frac{\pi}{\omega_i}\theta_i)\eta_i + \xi_i$, where $\xi_i \in H_0^1(\Omega)$. The dual singular solutions satisfy the following equation:

$$w_i^* \in L^2(\Omega) \setminus H_0^1(\Omega), \qquad -\Delta w_i^* = 0 \quad \text{in } \Omega, \qquad w_i^* = 0 \quad \text{on } \Gamma \setminus \{x_i\}.$$

If $f \in L^2(\Omega)$, the coefficient of singularity c_i at the vertex x_i, associated to the solution v of problem

$$\text{Find } v \in H_0^1(\Omega) \text{ such that } \quad -\Delta v = f \quad \text{in } \Omega,$$

is given by

$$c_i = \int_\Omega w_i^* f \,\mathrm{d}x. \tag{4}$$

Remark 4.1. The set $\{w_0^*,\ldots,w_m^*\}$ is linearly independent.

4.2. Cancellation of singularities by internal action

Theorem 4.1. *There exists $m+1$ $\mathcal{C}^\infty$ functions with compact support in ϖ, $g_0,\ldots,g_m$ such that if $f \in L^2(\Omega)$ and $c_0,\ldots,c_m$ are defined in* (4) *then the solution of problem*

$$\textit{Find } y \in H_0^1(\Omega) \textit{ such that } \quad -\Delta y = f - \sum_{i=0}^{m} c_i g_i \quad \textit{in } \Omega, \tag{5}$$

is in $H^2(\Omega)$.

Proof. The dual singular solutions w_i^* verify hypothesis of Theorem 3.1. Therefore, there exists $m+1$ $\mathcal{C}^\infty$ functions with compact support in ϖ, $g_0,\ldots,g_m$ such that:

$$\forall i, j \in \{0,\ldots,m\}, \qquad \int_\Omega w_i^* g_j \,\mathrm{d}x = \delta_{ij}.$$

Let $c_0, \ldots, c_m$ be the coefficients of singularity defined in (4). Then, the solution of problem (5) is in $H^2(\Omega)$. In fact the coefficients of singularity $\alpha_0, \ldots, \alpha_m$ associated to the solution of (5) are given by

$$\alpha_i = \int_\Omega w_i^* \left(f - \sum_{j=0}^m c_j g_j \right) \mathrm{d}x = \int_\Omega w_i^* f \,\mathrm{d}x - \sum_{j=0}^m \left[c_j \int_\Omega w_i^* g_j \,\mathrm{d}x \right].$$

Then, due to Theorem 3.1, it follows $\alpha_i = c_i - \sum_{j=0}^m c_j \delta_{ij} = 0$, and we conclude that $y \in H^2(\Omega)$. □

4.3. Cancellation of singularities by acting on Dirichlet conditions

Theorem 4.2. *There exists $m+1$ $\mathcal{C}^\infty$ functions with compact support in Γ_0, $h_0, \ldots, h_m$ such that if $f \in L^2(\Omega)$ and $c_0, \ldots, c_m$ are defined in* (4) *then the solution of problem*

$$\text{Find } y \in H^1(\Omega) \text{ such that } \quad -\Delta y = f \quad \text{in } \Omega, \quad y = \sum_{i=0}^m c_i h_i \quad \text{on } \Gamma, \tag{6}$$

is in $H^2(\Omega)$.

Proof. The dual singular solutions w_i^* verify hypothesis of Theorem 3.2. Therefore, there exists $m+1$ $\mathcal{C}^\infty$ functions with compact support in Γ_0, $h_0, \ldots, h_m$ such that

$$\forall i, j \in \{0, \ldots, m\}, \quad \int_\Gamma \frac{\partial w_i^*}{\partial \nu} h_j \,\mathrm{d}\sigma = \delta_{ij}.$$

Let z be an $\mathcal{C}^\infty$ extension of $\sum_{i=0}^m c_i h_i$ in Ω with support in a neighbourhood of Γ_0. Let $v = y - z$ then $v = 0$ on Γ and $-\Delta v = f + \Delta z$. Denote by $\beta_0, \ldots, \beta_m$ the coefficients of singularity associated to v. Then by integrating by parts over Ω, we obtain

$$\beta_i = \int_\Omega (f + \Delta z) w_i^* \,\mathrm{d}x = 0.$$

This allows us to conclude that $v \in H^2(\Omega)$, so $y \in H^2(\Omega)$. □

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