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Algebra

A construction of semisimple tensor categories

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Abstract

Let $\mathcal{A}$ be an Abelian category such that every object has only finitely many subobjects. From $\mathcal{A}$ we construct a semisimple tensor category $\mathcal{T}$. We show that $\mathcal{T}$ interpolates the categories $\operatorname{Rep}(\operatorname{Aut}(p), K)$ where p runs through certain projective pro-objects of $\mathcal{A}$. This extends a construction of Deligne for symmetric groups. ***To cite this article: F. Knop, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Une construction des catégories tensorielles semi-simples. Soit $\mathcal{A}$ une catégorie abélienne dont chaque objet n'a qu'un nombre fini de sous-objets. A partir de $\mathcal{A}$ on construit une catégorie tensorielle semi-simple $\mathcal{T}$. On démontre que $\mathcal{T}$ interpole les catégories $\operatorname{Rep}(\operatorname{Aut}(p), K)$ où p parcourt certains pro-objets projectifs de $\mathcal{A}$. Ceci étend une construction de Deligne pour les groupes symétriques. ***Pour citer cet article : F. Knop, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

1. Introduction

Let K be a field of characteristic zero. In [2], Deligne constructed a tensor category $\operatorname{Rep}(S_t, K)$ over K depending on a parameter $t \in K$. If $t \notin \mathbb{N}$ then $\operatorname{Rep}(S_t, K)$ is Abelian semisimple. Otherwise, it has as quotient the category of finite-dimensional representations of the symmetric group S_t.

In this Note, we extend Deligne's construction. Starting from an Abelian category $\mathcal{A}$ such that every object has only finitely many subobjects we construct a tensor category $\mathcal{T} = \mathcal{T}(\mathcal{A}, K)$ which depends on parameters t_φ, one for each isomorphism class of simple objects in $\mathcal{A}$. We show that $\mathcal{T}$ is semisimple if none of the parameters is singular (see Definition 3.1). Then the simple objects of $\mathcal{T}$ correspond to pairs (x, π) where x is an object of $\mathcal{A}$ and π is an irreducible representation of $\operatorname{Aut}_{\mathcal{A}}(x)$. If all parameters are singular, the category $\mathcal{T}$ has as quotient $\operatorname{Rep}(\operatorname{Aut}(p), K)$ where p is a projective (pro-)object of $\mathcal{A}$.

The main example is $\mathcal{A} = \operatorname{Mod}(\mathbb{F}_q)$, the category of finite-dimensional $\mathbb{F}_q$-vector spaces. In that case, the simple objects of $\mathcal{T}$ correspond to irreducible representations of $GL(m, \mathbb{F}_q)$, $m \in \mathbb{N}$. There is only one parameter and this

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parameter is singular if and only it is a power q^n. In that case, the category has as a quotient $\mathrm{Rep}(GL(n,\mathbb{F}_q),K)$. This proves conjecture [2], p. 3, (A) of Deligne. In unpublished work, Deligne has also proved his conjecture.

2. The construction of $\mathcal{T}(\mathcal{A},K)$

Let $\mathcal{A}$ be an essentially small Abelian category such that every object has finite length. Let $\hat{\mathcal{A}}$ be the set of isomorphism classes of simple objects of $\mathcal{A}$. The category we are going to construct will depend on $\mathcal{A}$ and on a fixed map $\hat{\mathcal{A}} \to K$ where K is a field. The image of $\varphi \in \hat{\mathcal{A}}$ in K will be denoted by t_φ. Let $\kappa(\mathcal{A})$ be the free commutative monoid generated by $\hat{\mathcal{A}}$. It coincides with the Grothendieck monoid of $\mathcal{A}$. Thus, every object x gives rise to an element $\langle x\rangle$ of $\kappa(\mathcal{A})$. In particular, $\langle x\rangle = \langle y\rangle$ if and only if x and y have the same composition factors.

A *correspondence* between two objects x and y is a morphism $F: c \to x \oplus y$. If F is a monomorphism then it is called a *relation*. Two correspondences $F: c \to x\oplus y$ and $G: d \to x \oplus y$ are called *equivalent* if

$$\operatorname{im} F = \operatorname{im} G \quad \text{and} \quad \langle \ker F\rangle = \langle \ker G\rangle. \tag{1}$$

We define the pseudo-Abelian tensor category $\mathcal{T}$ in several steps. First we define the category $\mathcal{T}_0$:

Objects: Same as $\mathcal{A}$. The object x of $\mathcal{A}$, regarded as an object of $\mathcal{T}_0$, will be denoted by $[x]$.
Morphisms: Equivalence classes of correspondences.
Composition: If $G: c \to x \oplus y$ and $F: c \to y \oplus z$ are correspondences then FG is the equivalence class of $c \times_y d \to x \oplus z$. It is easy to see that the composition is well defined and associative.

The category $\mathcal{T}_0$ becomes a symmetric monoidal category by defining $[x]\otimes[y] := [x \oplus y]$. The unit object is $\mathbf{1} = [0]$. Each object is selfdual with $\delta: \mathbf{1} \to [x]\otimes[x]$ and $\mathrm{ev}: [x]\otimes[x] \to \mathbf{1}$ given by the diagonal morphism $x \to x \oplus x$.

Let $\mathcal{T}_1(\mathcal{A})$ be the category with the same objects as $\mathcal{T}_0$ but with $\mathrm{Hom}_{\mathcal{T}_1(\mathcal{A})}([x],[y])$ being the free Abelian group generated by $\mathrm{Hom}_{\mathcal{T}_0}([x],[y])$. The ring $\mathrm{End}_{\mathcal{T}_1(\mathcal{A})}(\mathbf{1})$ is isomorphic to the polynomial ring $\mathbb{Z}[\hat{\mathcal{A}}]$. Thus, every Hom-space is a $\mathbb{Z}[\hat{\mathcal{A}}]$-module. The fixed map $\hat{\mathcal{A}} \to K$ induces a homomorphism $\mathbb{Z}[\hat{\mathcal{A}}] \to K$. Now we define the category $\mathcal{T}_1(\mathcal{A},K)$ as having the same objects as $\mathcal{T}_1(\mathcal{A})$ but with morphisms

$$\mathrm{Hom}_{\mathcal{T}_1(\mathcal{A},K)}\big([x],[y]\big) = \mathrm{Hom}_{\mathcal{T}_1(\mathcal{A})}\big([x],[y]\big) \otimes_{\mathbb{Z}[\hat{\mathcal{A}}]} K.$$

Finally, let $\mathcal{T} = \mathcal{T}(\mathcal{A},K)$ be the pseudo-Abelian completion of $\mathcal{T}_1(\mathcal{A},K)$, i.e., the category obtained by adjoining finite direct sums and images of idempotents. The tensor product on $\mathcal{T}_0$ induces a symmetric K-bilinear tensor product on $\mathcal{T}$ such that every object has a dual.

3. The semisimplicity of $\mathcal{T}(\mathcal{A},K)$ for regular parameters

We call $\mathcal{A}$ *finitary* if every object has only finitely many subobjects. In that case, all Hom-spaces of $\mathcal{A}$ are finite and all Hom-spaces of $\mathcal{T}(\mathcal{A},K)$ are finite-dimensional K-vector spaces. For $\varphi \in \hat{\mathcal{A}}$ let m_φ be a simple object in the isomorphism class φ. Then $\mathrm{End}_{\mathcal{A}}(m_\varphi)$ is a finite field. Let q_φ be its order.

Definition 3.1. An element $t \in K$ is *φ-singular* if either

(i) $t \in \{1, q_\varphi, q_\varphi^2, \ldots\}$ or
(ii) $t = 0$ and m_φ is part of a non-splitting short exact sequence.

Theorem 3.2. *Let $\mathcal{A}$ be an essentially small finitary Abelian category. Assume that K is a field of characteristic zero and that there is no $\varphi \in \hat{\mathcal{A}}$ such that t_φ is φ-singular. Then $\mathcal{T}(\mathcal{A},K)$ is a semisimple tensor category. The simple objects correspond to isomorphism classes of pairs (x,π) where x is an object of $\mathcal{A}$ and π is an irreducible representation (over K) of* $\mathrm{Aut}_{\mathcal{A}}(x)$.

The proof has two main ingredients.

Lemma 3.3. *The pairing* $\mathrm{Hom}_{\mathcal{T}}(\mathbf{1}, X) \times \mathrm{Hom}_{\mathcal{T}}(X, \mathbf{1}) \to K : (F, G) \mapsto \mathrm{tr}\, GF$ *is perfect for all* $X \in \mathrm{Ob}\,\mathcal{T}$.

Proof. It suffices to check this for $X = [x]$. Then the assertion boils down to the non-vanishing of the determinant $\Delta_x := \det(\langle u \cap v\rangle_K)_{u,v \subseteq x}$. Here $\langle u\rangle_K$ denotes the image of $\langle u\rangle \in \kappa(\mathcal{A}) \subseteq \mathbb{Z}[\hat{\mathcal{A}}]$ in K. A formula of Lindström [3] and Wilf [6] implies

$$\Delta_x = \prod_{y \subseteq x} p_y \quad \text{with } p_y := \sum_{u \subseteq y} \mu(u, y)\langle u\rangle_K. \tag{2}$$

Here $\mu(u, y)$ is the Möbius function of the subobject lattice of x (or y). Let m be a simple subobject of y. Then p_y factorizes as $p_y = (t_\varphi - \alpha)\, p_{y/m}$ where α is the number of complements of m in y (Stanley [4,5]). Since α is either 0 or a power of q_φ we conclude by induction. □

Now we come to the second main ingredient. Let $\ell(x)$ denote the length of an object x of $\mathcal{A}$.

Definition 3.4. For a $\mathcal{T}$-morphism F let $\ell(F)$ be the least number l such that F factorizes through $[x_1] \oplus \cdots \oplus [x_s]$ with $\ell(x_i) \leqslant l$ for all i.

For the next statement, note that $x \mapsto [x]$, $f \mapsto \mathrm{graph}(f)$ defines an embedding of $\mathcal{A}$ into $\mathcal{T}$.

Lemma 3.5. *Let x and y be two objects of $\mathcal{A}$ with $\ell(x) = \ell(y) = l$. Then*

$$\mathrm{Hom}_{\mathcal{T}}\big([x], [y]\big) = K\big[\mathrm{Isom}_{\mathcal{A}}(x, y)\big] \oplus \big\{F \in \mathrm{Hom}_{\mathcal{T}}\big([x], [y]\big) \mid \ell(F) < l\big\}. \tag{3}$$

Proof. For a correspondence $F : c \to x \oplus y$ with components $F_x : c \to x$ and $F_y : c \to y$ let

$$\mathrm{core}\, F := c/(\ker F_x + \ker F_y). \tag{4}$$

Then (3) can be deduced from the following claims:

(i) F factorizes in $\mathcal{T}_0$ through $[\mathrm{core}\, F]$.
(ii) If F factorizes in $\mathcal{T}_0$ through $[z]$ then $\ell(z) \geqslant \ell(\mathrm{core}\, F)$.
(iii) $F \in \mathrm{Isom}_{\mathcal{A}}(x, y)$ if and only if $\ell(\mathrm{core}\, F) = l$.

With these two propositions at hand, the proof of Theorem 3.2 proceeds along the same lines as that of [2], Theorem 2.18. □

4. Specialization of $\mathcal{T}(\mathcal{A}, K)$ at singular parameters

In this section we study the category $\mathcal{T}$ when all parameters t_φ are singular but not zero. More precisely assume $t_\varphi = q_\varphi^{r_\varphi}$ with $r_\varphi \in \mathbb{N}$ for all $\varphi \in \hat{\mathcal{A}}$. Let $\mathcal{A}_p$ be the category of all pro-objects of $\mathcal{A}$. This category has enough projectives. Let

$$p \twoheadrightarrow \bigoplus_{\varphi \in \hat{\mathcal{A}}} m_\varphi^{\oplus r_\varphi} \tag{5}$$

be a projective cover and put $A(p) := \mathrm{Aut}_{\mathcal{A}_p}(p)$, a profinite group. Let $\mathcal{N}(Y, X)$ be the set of all $\mathcal{T}$-morphisms $F : Y \to X$ with $\mathrm{tr}\, GF = 0$ for all $G : X \to Y$. Then $\mathcal{N}$ forms a tensor ideal of $\mathcal{T}$.

Theorem 4.1. *There is a functor* $\mathcal{S} : \mathcal{T} \to \mathrm{Rep}(A(p), K)$ *which identifies* $\mathcal{T}/\mathcal{N}$ *with* $\mathrm{Rep}(A(p), K)$.

Proof. For a set S let $K[S]$ be the space of all functions $S \to K$. First we define a functor $\mathcal{S} : \mathcal{T}_0 \to \mathrm{Rep}(A(p), K)$. Let x be an object of $\mathcal{A}$ and $F : c \to x \oplus y$ a relation. Then we define

$$\mathcal{S}\big([x]\big) := K\big[\mathrm{Hom}_{\mathcal{A}_p}(p, x)\big], \qquad \mathcal{S}(F) : \mathcal{S}\big([x]\big) \to \mathcal{S}\big([y]\big) : \alpha \mapsto \sum_{\substack{\beta : p \to c \\ \alpha = F_x \beta}} (F_y \beta). \tag{6}$$

Using the projectivity of p one checks that $\mathcal{S}$ is a well defined tensor functor. Next, (5) implies that $\mathcal{S}(x \to 0 \oplus 0) = \langle x \rangle_K \in K$. This ensures that $\mathcal{S}$ extends uniquely to $\mathcal{T}_1$ and then to $\mathcal{T}$. That $\mathcal{S}$ has the stated property follows along the same lines as the proof of [2], Theorem 6.2. □

5. Examples and final remarks

As for examples, we already mentioned $\mathcal{A} = \mathrm{Mod}(\mathbb{F}_q)$ in the introduction. We point out three more:

1. Let $\mathcal{A}$ be the category of homomorphisms $U \to V$ between $\mathbb{F}_q$-vector spaces. Then $\mathcal{T}$ interpolates the representations of the parabolic $\binom{*\ *}{0\ *} \subseteq GL(n_1 + n_2, \mathbb{F}_q)$ with arbitrary block sizes n_1 and n_2. The set $\hat{\mathcal{A}}$ consists of two elements. The regular parameters are $t_1, t_2 \neq 0, 1, q, q^2, \ldots.$
2. Let $\mathcal{A}$ be the category of pairs (V, α) where V is an $\mathbb{F}_q$-vector space and α is a nilpotent endomorphism of V. Then $\mathcal{T}$ interpolates $\mathrm{Rep}(GL(n, \mathbb{F}_q[[x]]), K)$ with regular values $t \neq 0, 1, q, q^2, \ldots.$
3. Let $\mathcal{A}$ be the category of all finite Abelian p-groups. Then $\mathcal{T}$ interpolates $\mathrm{Rep}(GL(n, \hat{\mathbb{Z}}_p), K)$ with regular values $t \neq 0, 1, p, p^2, \ldots.$

Remarks.

1. Deligne's category $\mathrm{Rep}(S_t, K)$ is obtained by taking for $\mathcal{A}$ the opposite of the category of finite sets. Of course, this category is not Abelian. However, most of the results work more generally in the framework of exact Mal'cev categories in the sense of [1]. These comprise not only all Abelian categories and the opposite of the category of sets but also the categories of finite groups, finite rings and many more. In particular, it is possible to interpolate the representation categories of wreath products $S_n \wr G$ (for fixed G) or $S_{n_1} \wr S_{n_2} \wr S_{n_3} \cdots$. Details will appear elsewhere.
2. The basic objects in [2] are slightly different. Let x be an object of $\mathcal{A}$. Then every subobject y of x gives rise to an idempotent via the relation $y \hookrightarrow x \oplus x$. These idempotents commute and induce a decomposition $[x] = \bigoplus_{y \subseteq x} [y]^*$. It is these $[y]^*$, Deligne is working with. Observe that $\mathcal{S}([x]^*)$ is the space of functions on the set of all *epimorphisms* $p \twoheadrightarrow x$. The factorization (2) is an easy consequence of the decomposition.

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