Available online at www.sciencedirect.com

SCIENCE DIRECT®

C. R. Acad. Sci. Paris, Ser. I 343 (2006) 63–68

COMPTES RENDUS MATHEMATIQUE

http://france.elsevier.com/direct/CRASS1/

Calculus of Variations

Nonexistence of Ginzburg–Landau minimizers with prescribed degree on the boundary of a doubly connected domain

Leonid Berlyand [a,1], Dmitry Golovaty [b,2], Volodymyr Rybalko [c,3]

[a] *Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA*
[b] *Department of Theoretical and Applied Mathematics, The University of Akron, Akron, OH 44325, USA*
[c] *Mathematical Division, B. Verkin Institute for Low Temperature Physics and Engineering, 47 Lenin Ave., 61164 Kharkov, Ukraine*

Received 23 January 2006; accepted after revision 15 May 2006

Presented by Haïm Brezis

Abstract

Let ω, Ω be bounded simply connected domains in $\mathbb{R}^2$, and let $\overline{\omega} \subset \Omega$. In the annular domain $A = \Omega \setminus \overline{\omega}$ we consider the class $\mathcal{J}$ of complex valued maps having modulus 1 and degree 1 on $\partial\Omega$ and $\partial\omega$.

We prove that, when $\operatorname{cap}(A) < \pi$, there exists a finite threshold value κ_1 of the Ginzburg–Landau parameter κ such that the minimum of the Ginzburg–Landau energy E_κ not attained in $\mathcal{J}$ when $\kappa > \kappa_1$ while it is attained when $\kappa < \kappa_1$. ***To cite this article: L. Berlyand et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Nonexistence des minimizers de Ginzburg–Landau avec le degré prescrit sur la frontière d'un domaine doublement connexe. Soient ω, Ω des ouverts bornés, simplement connexes de $\mathbb{R}^2$, et soit $\overline{\omega} \subset \Omega$. Dans le domaine annulaire $A = \Omega \setminus \overline{\omega}$ on considère une classe $\mathcal{J}$ des applications à valeurs complexes ayant le module égal à 1 et le degré 1 sur $\partial\Omega$ et $\partial\omega$.

On montre que, si $\operatorname{cap}(A) < \pi$, alors il existe une valeur critique finie κ_1 du paramètre κ de Ginzburg–Landau, telle que le minimum de l'énergie de Ginzburg–Landau E_κ n'est pas atteint dans $\mathcal{J}$ pour $\kappa > \kappa_1$, tandis qu'il est attaint pour $\kappa < \kappa_1$. ***Pour citer cet article : L. Berlyand et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

Soient ω, Ω des ouverts bornés, simplement connexes de $\mathbb{R}^2$, tels que $\overline{\omega} \subset \Omega$. Dans le domaine annulaire $A = \Omega \setminus \overline{\omega}$ on considère une classe $\mathcal{J}$ des applications à valeurs complexes ayant le module égal à 1 et le degré 1 sur $\partial\Omega$ et $\partial\omega$.

On montre que, si $\operatorname{cap}(A) < \pi$, alors il existe une valeur critique finie κ_1 du paramètre κ de Ginzburg–Landau, telle que le minimum m_κ de l'énergie de Ginzburg–Landau E_κ n'est pas atteint dans $\mathcal{J}$ pour $\kappa > \kappa_1$, tandis qu'il est atteint pour $\kappa < \kappa_1$.

E-mail addresses: berlyand@math.psu.edu (L. Berlyand), dmitry@math.uakron.edu (D. Golovaty), vrybalko@ilt.kharkov.ua (V. Rybalko).

[1] Supported by the NSF grant DMS-0204637.
[2] Supported by the NSF grant DMS-0407361.
[3] Supported by the grant GP/F8/0045 Φ8/308-2004.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2006.05.013

La démonstration du résultat essentiel (voir Théorème 1.1) est basée sur l'éstimation $m_\kappa \leqslant 2\pi$ obtenue dans [7] ainsi que sur des résultats de convergence connus de [6].

On procède par contradiction. Supposons que le Théorème 1.1 est faux. Alors le minimiseur de E_κ existe pour tout κ finis.

On suppose d'abord que $\mathcal{A}$ est un anneau circulaire conforme équivalent à A. Sous l'hypothèse ci-déssus on montre que le minimum de l'énergie de Ginzburg–Landau pour le domaine $\mathcal{A}$ est atteint dans la classe $\mathcal{J}$ pour chaque κ. On peut supposer alors que A est un anneau circulaire.

Puisque m_κ est atteint pour tous κ il existe $u_\kappa \in \mathcal{J}$ tel que $E_\kappa[u_\kappa] = m_\kappa \leqslant 2\pi$.

Ensuite on construit une famille de fonctionnelles quadratiques auxiliaires $\{F_\kappa\}_{\kappa>0}$ sur un domaine rectangulaire. Sur la base de la famille $\{u_\kappa\}_{\kappa>0}$ on construit une famille de fonctions v_κ telles que $F_\kappa[v_\kappa] \leqslant 2\pi$.

On trouve alors des solutions explicites w_κ d'un système de EDP linéaires de Euler–Langrange qui corespond à la fonctionnelle F_κ, et on montre que $F_\kappa[w_\kappa] > 2\pi$. Ce qui mène à l'inégalité $F_\kappa[v_\kappa] \geqslant F_\kappa[w_\kappa] > 2\pi$ et achève la preuve par l'absurde.

1. Introduction

The present Note establishes nonexistence of minimizers of the Ginzburg–Landau functional in a class of Sobolev functions with prescribed degree on the boundary of an annular domain when the H^1-capacity of the domain is less than the critical value $c_{\mathrm{cr}} = \pi$. Here an annular domain is any domain in $\mathbb{R}^2$ conformally equivalent to a circular annulus.

Consider the minimization problem for the Ginzburg–Landau functional

$$E_\kappa[u] = \frac{1}{2}\int_A |\nabla u|^2\,\mathrm{d}x + \frac{\kappa^2}{4}\int_A \bigl(|u|^2 - 1\bigr)^2\,\mathrm{d}x \to \inf, \quad u \in \mathcal{J}, \tag{1}$$

where $A = \Omega \setminus \overline{\omega}$, $\overline{\omega} \subset \Omega$, and ω, Ω are bounded, simply connected domains in $\mathbb{R}^2$ with smooth boundaries. The class $\mathcal{J}$ is defined by

$$\mathcal{J} = \bigl\{u \in H^1(A)\colon\ |u| = 1 \text{ on } \partial\Omega \cup \partial\omega;\ \deg(u, \partial\Omega) = \deg(u, \partial\omega) = 1\bigr\}. \tag{2}$$

Note that a minimizer of (1) in $\mathcal{J}$ satisfies the Ginzburg–Landau equation

$$-\Delta u + \kappa^2\bigl(|u|^2 - 1\bigr)u = 0 \quad \text{in } A, \tag{3}$$

along with the natural boundary conditions $\frac{\partial u}{\partial \nu} \times u = 0$ on ∂A.

Problem (1) originates with a Ginzburg–Landau variational model of superconducting persistent currents in multiply connected domains.

The asymptotics as $\kappa \to \infty$ of global minimizers for the Ginzburg–Landau functional and their vortex structure for the Dirichlet boundary data (for which the degree is fixed by default) were studied in detail in [8] for simply-connected domains. For multiply connected domains and the Ginzburg–Landau functional with a magnetic field, the existence of local minimizers for large κ was established in [11] (cf. [10]) while the existence and properties of *global* minimizers was studied in [2].

The variational problems for the Ginzburg–Landau functional in a class of maps with the degree boundary conditions were considered in [7,9,4–6]. The difficulty in establishing the existence of minimizers over the class $\mathcal{J}$ is due to the fact that $\mathcal{J}$ is not closed with respect to weak H^1-topology [6] and one cannot use the direct method of calculus of variations. For a narrow circular annulus both existence and uniqueness were proved in [9] for an *arbitrary* (not necessarily large) $\kappa > 0$. In [4–6], a general approach for arbitrary multiply connected domains was developed. The existence of critical H^1-capacity, $c_{\mathrm{cr}} = \pi$, was established—the minimizers were shown always to exist when a domain has the capacity $\mathrm{cap}(A)$ that exceeds π. When $\mathrm{cap}(A)$ is below π, the minimizers of E_κ were shown to exist only when κ is small. Further it was conjectured in [4–6] that, when $\mathrm{cap}(A) < \pi$, there exists a threshold value κ_1 such that the minimum of E_κ is *not attained* when $\kappa > \kappa_1$ and it is attained when $\kappa < \kappa_1$.

The existence of κ_1 is established in this Note via the following:

Theorem 1.1. *Let $m_\kappa = \mathrm{Inf}\{E_\kappa[u],\ u \in \mathcal{J}\}$. Assume $\mathrm{cap}(A) < \pi$. Then there is a finite $\kappa_1 > 0$ such that m_κ is always attained for $\kappa < \kappa_1$ and it is never attained for $\kappa > \kappa_1$.*

2. Proof of Theorem 1.1

We argue by contradiction. Suppose that for all $\kappa > 0$, the infimum m_κ is attained at some map $u_\kappa \in \mathcal{J}$. Then $E_\kappa[u_\kappa] \leqslant 2\pi$, since $m_\kappa \leqslant 2\pi$ for all $\kappa > 0$ [7].

Next, we show that, without loss of generality, we can assume that A is a circular annulus $A = \{x \in \mathbb{R}^2\colon\ R > |x| > \frac{1}{R}\}$.

2.1. Conformal equivalence to a circular annulus

Proposition 2.1. *Suppose that A is such that m_κ is attained for every $\kappa > 0$. Then the same holds for the annular domain $\mathcal{A} := \{x\colon \exp(-\frac{\pi}{\mathrm{cap}(A)}) < |x| < \exp(\frac{\pi}{\mathrm{cap}(A)})\}$, where $\mathcal{A}$ is conformally equivalent to A.*

Proof. First, observe [1] that A is conformally equivalent to a circular annulus $\mathcal{A}$; moreover the corresponding conformal map $\mathcal{F}$ extends to a C^1-diffeomorphism of $\bar{A}$ onto $\bar{\mathcal{A}}$ that preserves the orientation of curves.

Given a $\kappa > 0$, let u_κ be a minimizer of the functional $E_\kappa[u]$ in $\mathcal{J}$. Then $E_\kappa[u_\kappa] < 2\pi$. Indeed, for any $\kappa' > \kappa$ there is a minimizer $u_{\kappa'}$ of $E_{\kappa'}[u]$ in $\mathcal{J}$ and $E_{\kappa'}[u_{\kappa'}] \leqslant 2\pi$, since $m_\kappa \leqslant 2\pi$ for all $\kappa > 0$ [7]. Then

$$E_{\kappa'}[u_{\kappa'}] - E_\kappa[u_\kappa] \geqslant E_{\kappa'}[u_{\kappa'}] - E_\kappa[u_{\kappa'}] = \frac{(\kappa')^2 - \kappa^2}{4} \int_A \left(|u_{\kappa'}|^2 - 1\right)^2 \mathrm{d}x,$$

so that $m_\kappa \leqslant 2\pi$ and $m_\kappa = 2\pi$ if and only if $|u_{\kappa'}| = 1$ a.e. in A. Since $u_{\kappa'}$ is a solution of Ginzburg–Landau equation (3)—$u_{\kappa'}$ minimizes $E_{\kappa'}[u]$ with respect to its own boundary data—the pointwise equality $|u_{\kappa'}| = 1$ a.e. in A implies that $u_{\kappa'} \equiv \mathrm{const}$. This is impossible since $u_{\kappa'} \in \mathcal{J}$ and we arrive at $E_\kappa[u_\kappa] < 2\pi$.

By using the conformal change of variables $x \to \mathcal{F}(x)$ and since $E_\kappa[u_\kappa] < 2\pi$, we obtain that

$$\frac{1}{2}\int_{\mathcal{A}} |\nabla\tilde{u}|^2\,\mathrm{d}x + \frac{\kappa^2}{4}\int_{\mathcal{A}} \left(|\tilde{u}|^2 - 1\right)^2 \mathrm{Jac}\left(\mathcal{F}^{-1}\right)\mathrm{d}x < 2\pi,$$

where $\tilde{u}(x) = u_\kappa(\mathcal{F}^{-1}(x))$. Since κ is arbitrary and m_κ is attained whenever $m_\kappa < 2\pi$ [6], we obtain that the minimum of (1) is attained for all $\kappa > 0$.

Note that, by the definition of H^1-capacity and conformal invariance of the Dirichlet integral, we have that $\mathrm{cap}(\mathcal{A}) = \mathrm{cap}(A)$. $\square$

2.2. Reduction to a linear problem

We make use of the following theorem from [4]:

Theorem 2.2. *Let* $\mathrm{cap}(A) < \pi$, *and suppose that* $u_\kappa \in \mathcal{J}$ *is a solution of Ginzburg–Landau equation* (3) *such that* $E_\kappa(u_\kappa) < 2\pi + \mathrm{e}^{-\kappa}$. *Then there is* $\gamma_\kappa = \mathrm{const} \in S^1$ *such that for any compact set* K *in* A

$$\|u_\kappa - \gamma_\kappa\|_{C^l(K)} = \mathrm{o}\left(\kappa^{-m}\right), \quad \text{as } \kappa \to \infty,\ \forall m > 0,\ l \in \mathbb{N}, \tag{4}$$

$$\int_A \left(|u_\kappa|^2 - 1\right)^2 \mathrm{d}x = \mathrm{o}\left(\kappa^{-m}\right), \quad \text{as } \kappa \to \infty,\ \forall m > 0. \tag{5}$$

Multiplying Eq. (3) by $\log\frac{|x|}{R}$ and integrating over $D = \{x\colon\ 1 < |x| < R\}$ we obtain

$$\begin{aligned} 0 &= \int_D \Delta u_\kappa \log\frac{|x|}{R}\,\mathrm{d}x + \kappa^2 \int_D u_\kappa\left(1 - |u_\kappa|^2\right)\log\frac{|x|}{R}\,\mathrm{d}x \\ &= \int_{\partial D} \frac{\partial u_\kappa}{\partial \nu}\log\frac{|x|}{R}\,\mathrm{d}\sigma - \int_{\partial D} u_\kappa \frac{\partial \log|x|}{\partial \nu}\,\mathrm{d}\sigma + \kappa^2 \int_D u_\kappa\left(1 - |u_\kappa|^2\right)\log\frac{|x|}{R}\,\mathrm{d}x \end{aligned}$$

$$= -\frac{1}{R}\int\limits_{|x|=R} u_\kappa\,\mathrm{d}\sigma + \int\limits_{|x|=1} u_\kappa\,\mathrm{d}\sigma$$
$$+ \int\limits_{|x|=1} \frac{\partial u_\kappa}{\partial \nu}\log\frac{1}{R}\,\mathrm{d}\sigma + \kappa^2\int\limits_D u_\kappa\big(1-|u_\kappa|^2\big)\log\frac{|x|}{R}\,\mathrm{d}x.$$

Using a similar calculation for $D=\{x\colon\ R^{-1}<|x|<1\}$ and (4)–(5), we have for large κ that

$$\frac{1}{R}\int\limits_{|x|=R} u_\kappa\,\mathrm{d}\sigma = 2\pi\gamma_\kappa + \mathrm{o}\big(\kappa^{-m}\big) \quad \text{and} \quad R\int\limits_{|x|=1/R} u_\kappa\,\mathrm{d}\sigma = 2\pi\gamma_\kappa + \mathrm{o}\big(\kappa^{-m}\big). \tag{6}$$

After the conformal change of variables $x \to (r,\varphi): x = \mathrm{e}^{r+\mathrm{i}\varphi}$, the functional E_κ takes the form

$$E_\kappa[u_\kappa] = \frac{1}{2}\int\limits_{-L}^{L}\mathrm{d}r\int\limits_0^{2\pi}\mathrm{d}\varphi\,|\nabla u_\kappa|^2 + \frac{\kappa^2}{4}\int\limits_{-L}^{L}\mathrm{e}^{2r}\,\mathrm{d}r\int\limits_0^{2\pi}\mathrm{d}\varphi\,\big(|u_\kappa|^2-1\big)^2,$$

where the integrals are taken over the rectangular region $-\log R<r<\log R$, $0\leqslant\varphi<2\pi$, and $L=\log R$.

We define $u_\kappa^{(1)}$ by setting

$$u_\kappa^{(1)}(r,\varphi) := \bar\gamma_\kappa\begin{cases} u_\kappa(r,\varphi), & 0\leqslant r<L,\\ u_\kappa(-r,\varphi), & -L<r<0,\end{cases} \quad \text{or} \quad u_\kappa^{(1)}(r,\varphi) := \bar\gamma_\kappa\begin{cases} u_\kappa(-r,\varphi), & 0\leqslant r<L,\\ u_\kappa(r,\varphi), & -L<r<0,\end{cases}$$

to obtain that $u_\kappa^{(1)}(r,\varphi)=u_\kappa^{(1)}(-r,\varphi)$ and

$$\frac{1}{2}\int\limits_{-L}^{L}\mathrm{d}r\int\limits_0^{2\pi}\mathrm{d}\varphi\,\big|\nabla u_\kappa^{(1)}\big|^2 + \frac{\kappa^2}{4R^2}\int\limits_{-L}^{L}\mathrm{d}r\int\limits_0^{2\pi}\mathrm{d}\varphi\,\big(\big|u_\kappa^{(1)}\big|^2-1\big)^2 \leqslant 2\pi. \tag{7}$$

Due to (4), for all $0<\rho<L$ and $m\in\mathbb{N}$, we have that

$$\max_{-\rho<r<\rho}\big|u_\kappa^{(1)}-1\big| = \mathrm{o}\big(\kappa^{-m}\big), \quad \text{as } \kappa\to\infty. \tag{8}$$

Next, we multiply $u_\kappa^{(1)}$ by a suitable constant of magnitude 1 and use (6) to introduce $u_\kappa^{(2)}$ so that, in addition to (7) and (8), it satisfies

$$\operatorname{Im}\int\limits_0^{2\pi} u_\kappa^{(2)}(L,\varphi)\,\mathrm{d}\varphi = \operatorname{Im}\int\limits_0^{2\pi} u_\kappa^{(2)}(-L,\varphi)\,\mathrm{d}\varphi = 0. \tag{9}$$

Observe that

$$\big(\big|u_\kappa^{(2)}\big|^2-1\big)^2 \geqslant \big(\mathrm{Re}\big(u_\kappa^{(2)}\big)-1\big)^2\big(\mathrm{Re}\big(u_\kappa^{(2)}\big)+1\big)^2 - 4\big(1-\mathrm{Re}\big(u_\kappa^{(2)}\big)\big)\big(\mathrm{Im}\big(u_\kappa^{(2)}\big)\big)^2.$$

Let

$$F_\kappa[w] := \frac{1}{2}\int\limits_{-L}^{L}\mathrm{d}r\int\limits_0^{2\pi}\mathrm{d}\varphi\,|\nabla w|^2 + \int\limits_{-\rho}^{\rho}\mathrm{d}r\int\limits_0^{2\pi}\mathrm{d}\varphi\left(\frac{\kappa^2}{2}\big(\mathrm{Re}(w)-1\big)^2 - \frac{\kappa^{-2}}{2}\big(\mathrm{Im}(w)\big)^2\right).$$

Then, using (7) and (8), one can show that for a sufficiently large $\kappa>0$ there exists a $\kappa'\geqslant\kappa$ such that the function $v_\kappa := u_{\kappa'}^{(2)}$ satisfies $F_\kappa[v_\kappa]\leqslant 2\pi$. Moreover, $|v_\kappa|=1$ as $r=\pm L$, the function v_κ is 2π-periodic in φ, and

$$v_\kappa = a_0^\kappa + \sum_{n=1}^{\infty}\big(a_n^\kappa\cos n\varphi + b_n^\kappa\sin n\varphi\big), \quad \text{as } r=\pm L.$$

In view of (9), we have that $\mathrm{Im}(a_0^\kappa)=0$. Further, by degree formula [4],

$$1 = \frac{1}{2i}\sum_{n=1}^{\infty} n\big(b_n^\kappa\bar a_n^\kappa - a_n^\kappa\bar b_n^\kappa\big) = \sum_{n=1}^{\infty} n\big(\mathrm{Re}\big(a_n^\kappa\big)\,\mathrm{Im}\big(b_n^\kappa\big) - \mathrm{Re}\big(b_n^\kappa\big)\,\mathrm{Im}\big(a_n^\kappa\big)\big). \tag{10}$$

For large κ there is a unique minimizer w_κ of $F_\kappa[w]$ in the class of functions 2π-periodic in φ and satisfying $w_\kappa = v_\kappa$ when $r = \pm L$. Then

$$F_\kappa[w_\kappa] \leqslant F_\kappa[v_\kappa] \leqslant 2\pi, \tag{11}$$

where w_κ is the solution of the *linear* (inhomogeneous) problem

$$\begin{cases} -\Delta \operatorname{Re}(w) + \kappa^2 V(r)\big(\operatorname{Re}(w) - 1\big) = 0, & -L < r < L, \\ -\Delta \operatorname{Im}(w) - \kappa^{-2} V(r) \operatorname{Im}(w) = 0, & -L < r < L, \\ w(r,\varphi) = w(r,\varphi + 2\pi), \\ w = v_\kappa, \quad r = \pm L. \end{cases} \tag{12}$$

Here $V(r) = 1$ when $-\rho < r < \rho$ and $V(r) = 0$ otherwise.

2.3. Energy estimate for problem (12)

Problem (12) has the unique solution for large κ in the form

$$w_\kappa(r,\varphi) = 1 + \big(a_0^\kappa - 1\big) w_{\kappa,0}^{(1)}(r) + \sum_{n=1}^{\infty} w_{\kappa,n}^{(1)}(r)\big(\operatorname{Re}\big(a_n^\kappa\big)\cos n\varphi + \operatorname{Re}\big(b_n^\kappa\big)\sin n\varphi\big) + \mathrm{i}\sum_{n=1}^{\infty} w_{\kappa,n}^{(2)}(r)\big(\operatorname{Im}\big(a_n^\kappa\big)\cos n\varphi + \operatorname{Im}\big(b_n^\kappa\big)\sin n\varphi\big)$$

with real-valued $w_{\kappa,n}^{(1)}$ and $w_{\kappa,n}^{(2)}$ (since a_0^κ is real). Functions $w_{\kappa,n}^{(1)}$, $w_{\kappa,n}^{(2)}$ can be found explicitly so that

$$F_\kappa[w_\kappa] = P_0^\kappa + \pi \sum_{n=1}^{\infty} n\Big(P_n^\kappa\big(\big|\operatorname{Re}\big(a_n^\kappa\big)\big|^2 + \big|\operatorname{Re}\big(b_n^\kappa\big)\big|^2\big) + Q_n^\kappa\big(\big|\operatorname{Im}\big(a_n^\kappa\big)\big|^2 + \big|\operatorname{Im}\big(b_n^\kappa\big)\big|^2\big)\Big). \tag{13}$$

Here $P_0^\kappa \geqslant 0$ while

$$P_n^\kappa = \frac{1 - \mathrm{e}^{-2n(L-\rho)} + (1 + \mathrm{e}^{-2n(L-\rho)})\sqrt{1 + \kappa^2 n^{-2}}\tanh(\rho\sqrt{n^2 + \kappa^2})}{1 + \mathrm{e}^{-2n(L-\rho)} + (1 - \mathrm{e}^{-2n(L-\rho)})\sqrt{1 + \kappa^2 n^{-2}}\tanh(\rho\sqrt{n^2 + \kappa^2})},$$

and

$$Q_n^\kappa = \frac{1 - \mathrm{e}^{-2n(L-\rho)} + (1 + \mathrm{e}^{-2n(L-\rho)})\sqrt{1 - (\kappa n)^{-2}}\tanh(\rho\sqrt{n^2 - \kappa^{-2}})}{1 + \mathrm{e}^{-2n(L-\rho)} + (1 - \mathrm{e}^{-2n(L-\rho)})\sqrt{1 - (\kappa n)^{-2}}\tanh(\rho\sqrt{n^2 - \kappa^{-2}})},$$

(see [3] for the detailed calculation). Then

$$F_\kappa[w_\kappa] \geqslant 2\pi \sum_{n=1}^{\infty} n\sqrt{P_n^\kappa Q_n^\kappa}\big(\big|\operatorname{Re}\big(a_n^\kappa\big)\big|\big|\operatorname{Im}\big(b_n^\kappa\big)\big| + \big|\operatorname{Re}\big(b_n^\kappa\big)\big|\big|\operatorname{Im}\big(a_n^\kappa\big)\big|\big). \tag{14}$$

Now we show that there exists a $\kappa_0 > 0$ such that for all $\kappa \geqslant \kappa_0$ and for all $n \geqslant 1$ we have

$$P_n^\kappa Q_n^\kappa > 1. \tag{15}$$

Indeed, we can rewrite P_n^κ and Q_n^κ as follows

$$P_n^\kappa = \frac{1 + \beta_n^\kappa \mathrm{e}^{-2n(L-\rho)}}{1 - \beta_n^\kappa \mathrm{e}^{-2n(L-\rho)}}, \quad \text{and} \quad Q_n^\kappa = \frac{1 - \alpha_n^\kappa \mathrm{e}^{-2n(L-\rho)}}{1 + \alpha_n^\kappa \mathrm{e}^{-2n(L-\rho)}},$$

where

$$\alpha_n^\kappa = \frac{1 - \sqrt{1 - (\kappa n)^{-2}}\tanh(\rho\sqrt{n^2 - \kappa^{-2}})}{1 + \sqrt{1 - (\kappa n)^{-2}}\tanh(\rho\sqrt{n^2 - \kappa^{-2}})} \quad \text{and} \quad \beta_n^\kappa = \frac{\sqrt{1 + \kappa^2 n^{-2}}\tanh(\rho\sqrt{n^2 + \kappa^2}) - 1}{\sqrt{1 + \kappa^2 n^{-2}}\tanh(\rho\sqrt{n^2 + \kappa^2}) + 1}.$$

Note that (15) is equivalent to the inequality $\alpha_n^\kappa < \beta_n^\kappa$. This inequality clearly holds for any fixed $n \geqslant 0$ when κ is sufficiently large, since $\alpha_n^\kappa \to \mathrm{e}^{-2n\rho}$, $\beta_n^\kappa \to 1$, as $\kappa \to \infty$. On the other hand, for large n and all $\kappa \geqslant 1$, we have

$\alpha_n^\kappa \leqslant \mathrm{e}^{-n\rho} + \frac{1}{(n\kappa)^2}$, $\beta_n^\kappa \geqslant \frac{\gamma}{n^2}$, where $\gamma > 0$ is independent of n and κ. Thus (15) and, therefore, (14) are satisfied once κ_0 is chosen to be sufficiently large.

By (15) and (14) we get

$$F_\kappa[w_\kappa] \geqslant 2\pi \sum_{n=1}^{\infty} n\left(\left|\mathrm{Re}\left(a_n^\kappa\right)\right|\left|\mathrm{Im}\left(b_n^\kappa\right)\right| + \left|\mathrm{Re}\left(b_n^\kappa\right)\right|\left|\mathrm{Im}\left(a_n^\kappa\right)\right|\right),$$

and by (15) this inequality is strict unless r.h.s. $= 0$. By (10)

$$\sum_{n=1}^{\infty} n\left(\left|\mathrm{Re}\left(a_n^\kappa\right)\right|\left|\mathrm{Im}\left(b_n^\kappa\right)\right| + \left|\mathrm{Re}\left(b_n^\kappa\right)\right|\left|\mathrm{Im}\left(a_n^\kappa\right)\right|\right) \geqslant 1,$$

so that $F_\kappa[w_\kappa] > 2\pi$. This contradicts (11).

References

[1] L. Ahlfors, Complex Analysis, McGraw-Hill, 1966.

[2] S. Alama, L. Bronsard, Vortices and pinning effects for the Ginzburg–Landau model in multiply connected domains, Comm. Pure Appl. Math. 59 (1) (2006) 36–70.

[3] L. Berlyand, D. Golovaty, V. Rybalko, Capacity of a multiply-connected domain and nonexistence of Ginzburg–Landau minimizers with prescribed degrees on the boundary, arXiv:math.AT/9910001, 2006. Preprint available at http://www.citebase.org/cgi-bin/citations?id=oai:arXiv.org:math/0601018.

[4] L. Berlyand, P. Mironescu, Ginzburg–Landau minimizers in perforated domains with prescribed degrees. Preprint available at http://desargues.univ-lyon1.fr.

[5] L. Berlyand, P. Mironescu, Ginzburg–Landau minimizers with prescribed degrees. Capacity of the domain and emergence of vortices, J. Funct. Anal. (2006), doi:10.1016/j:jfa.2006.03.006.

[6] L. Berlyand, P. Mironescu, Ginzburg–Landau minimizers with prescribed degrees: dependence on domain, C. R. Math. Acad. Sci. Paris 337 (2003) 375–380.

[7] L.V. Berlyand, K. Voss, Symmetry breaking in annular domains for a Ginzburg–Landau superconductivity model, in: Proceedings of IUTAM 99/4 Symposium, Sydney, Australia, Kluwer Academic Publishers, 1999, pp. 189–210.

[8] F. Bethuel, H. Brezis, F. Hélein, Ginzburg–Landau Vortices, Birkhäuser, 2004.

[9] D. Golovaty, L. Berlyand, On uniqueness of vector-valued minimizers of the Ginzburg–Landau functional in annular domains, Calc. Var. Partial Differential Equations 14 (2) (2002) 213–232.

[10] S. Jimbo, Y. Morita, Ginzburg–Landau equations and stable solutions in a rotational domain, SIAM J. Math. Anal. 27 (5) (1996) 1360–1385.

[11] J. Rubinstein, P. Sternberg, Homotopy classification of minimizers of the Ginzburg–Landau energy and the existence of permanent currents, Comm. Math. Phys. 179 (1) (1996) 257–263.