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C. R. Acad. Sci. Paris, Ser. I 343 (2006) 367–370
COMPTES RENDUS MATHEMATIQUE
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Calculus of Variations

Simple proof of two-well rigidity

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Received 13 March 2006; accepted after revision 4 July 2006

Available online 25 July 2006

Presented by John M. Ball

Abstract

We give a short proof of the rigidity estimate of Müller and Chaudhuri for two strongly incompatible wells. Making strong use of the arguments of Ball and James our approach shows that incompatibility for gradient Young measures can be used to reduce rigidity estimates for several wells to one-well rigidity. ***To cite this article: C. De Lellis, L. Székelyhidi Jr., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Une démonstration simple d'une estimation de rigidité pour deux puits. Nous donnons une démonstration simple d'une estimation de rigidité de Müller et Chaudhuri pour deux puits fortement incompatibles. Nous employons un argument de Ball et James pour montrer que l'incompatibilité pour les mesures de Young engendrées par des gradients permet de réduire les estimations de rigidité pour plusieurs puits à celles pour un puit. ***Pour citer cet article : C. De Lellis, L. Székelyhidi Jr., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

1. Introduction

A crucial ingredient in rigorous derivations of plate theories from three-dimensional elasticity [3,6] is a quantitative rigidity estimate in terms of the bulk energy of deformations close to zero-energy configurations. In nonlinear elasticity one usually considers sets of the form $K=\bigcup_{i=1}^m \mathrm{SO}(3)A_i$ as the set of deformations which carry zero bulk energy. The different copies of $\mathrm{SO}(n)$ are called energy wells. In the rigid situation, when the only deformations with zero energy (i.e. maps $u:\mathbb{R}^3\to\mathbb{R}^3$ satisfying $\nabla u\in K$) are affine maps, it is of interest to find estimates on the precise rate of convergence of approximating sequences. The starting point of such an analysis is the rigidity estimate of Friesecke, James and Müller [6], which says that for any Lipschitz domain $\Omega\subset\mathbb{R}^n$ and for any $p\in\,]1,\infty[$, there exists a constant $C(p,\Omega)$ so that

$$\inf_{R\in\mathrm{SO}(n)}\|\nabla u-R\|_{L^p(\Omega)}\leqslant C(p,\Omega)\big\|\mathrm{dist}(\nabla u,K)\big\|_{L^p(\Omega)}\quad\text{for all } u\in W^{1,p}\big(\Omega,\mathbb{R}^n\big), \tag{1}$$

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2006.07.008

where $K = \mathrm{SO}(n)$ (Friesecke, James and Müller proved in [6] only the case $p = 2$ of (1), but the corresponding inequalities for $p \in]1, \infty[$ can be obtained by minor modifications of the arguments, see for instance Section 2.4 of [5]).

Building on the methods developed in [6], Chaudhuri and Müller in [2] obtained the corresponding rigidity estimate for the case of two *strongly incompatible wells* $K = \mathrm{SO}(n)A_1 \cup \mathrm{SO}(n)A_2$. An important ingredient in the proof of Chaudhuri and Müller is the result of Matos in [8] that under certain conditions on the matrices A_1 and A_2 the exact solutions of the inclusion problem $\nabla u \in K$ are solutions of a certain strongly elliptic system. We also note that Matos used this observation in [8] to deduce incompatibility for gradient Young measures in the sense of Definition 1.1 below.

Our aim in this Note is to give a simple proof of how under the condition of incompatibility for gradient Young measures the rigidity estimate of the two-well problem reduces to the rigidity estimate [6] for the one-well problem. Our argument is very much based on the unpublished but well-known argument of Ball and James [1] for obtaining a transition-layer estimate for approximate solutions to differential inclusions with incompatible wells. Indeed, our estimate in Theorem 1.2 is very similar in spirit to the transition-layer estimate. In what follows we will use the shorthand notation $d_K^p(\cdot)$ for $(\mathrm{dist}(K, \cdot))^p$.

Definition 1.1. Let $K_1, K_2 \subset \mathbb{R}^{m\times n}$ be disjoint compact sets. We say that K_1, K_2 are *incompatible* for gradient Young measures if whenever ν_x is a gradient Young measure on some connected domain Ω such that $\operatorname{supp} \nu_x \subset K_1 \cup K_2$ for almost every $x \in \Omega$, then

$$\operatorname{supp} \nu_x \subset K_1 \quad \text{for a.e. } x \in \Omega \quad \text{or} \quad \operatorname{supp} \nu_x \subset K_2 \quad \text{for a.e. } x \in \Omega.$$

Theorem 1.2. *Suppose $K_1, K_2 \subset \mathbb{R}^{m\times n}$ are disjoint compact sets which are incompatible for gradient Young measures and let $K = K_1 \cup K_2$. Let $p \in [1, \infty[$ and $\Omega \subset \mathbb{R}^n$ be a connected Lipschitz domain. Then there exists a constant $C = C(p, \Omega)$ such that*

$$\min\left(\int_\Omega d_{K_1}^p(\nabla u)\,\mathrm{d}x, \int_\Omega d_{K_2}^p(\nabla u)\,\mathrm{d}x\right) \leqslant C(p, \Omega) \int_\Omega d_K^p(\nabla u)\,\mathrm{d}x \quad \textit{for all } u \in W^{1,p}\big(\Omega, \mathbb{R}^m\big). \tag{2}$$

Remark 1. Note that (2) holds even in the critical case $p = 1$, in contrast with estimate (1) (see [4]).

Proof. By a truncation argument it suffices to prove the inequality for maps with $\|\nabla u\|_{L^\infty(\Omega)} \leqslant M$ for some constant M depending on Ω, K and p. Indeed, since K is compact, we can choose positive constants R and C such that $|A| \leqslant C d_K(A)$ for every $A \in \mathbb{R}^{m\times n}$ with $|A| \geqslant R$. By Proposition A.1 of [6] there exists a constant $C = C(p, \Omega)$ such that, for every $v \in W^{1,p}(\Omega)$ there exists $u \in W^{1,\infty}(\Omega)$ satisfying the following properties:

(i) $\|\nabla u\|_{L^\infty} \leqslant CR$;
(ii) $|\{x \in \Omega\colon u(x) \neq v(x)\}| \leqslant CR^{-p} \int_{\{x\in\Omega\colon |\nabla v|(x)>R\}} |\nabla v|^p\,\mathrm{d}x$;
(iii) $\|\nabla v - \nabla u\|_{L^p}^p \leqslant C \int_{\{x\in\Omega\colon |\nabla v|(x)>R\}} |\nabla v|^p\,\mathrm{d}x$.

Recall that $d_K(A) \leqslant d_{K_i}(A) \leqslant C(1 + |A|)$. Hence, from (i)–(iii) and the choice of R, it follows easily that $\int d_K^p(\nabla u)\,\mathrm{d}x \leqslant C \int d_K^p(\nabla v)\,\mathrm{d}x$ and $\int d_{K_i}^p(\nabla v)\,\mathrm{d}x \leqslant C \int d_{K_i}^p(\nabla u)\,\mathrm{d}x + C \int d_K^p(\nabla v)\,\mathrm{d}x$. These inequalities show that it suffices to prove (2) for functions u which enjoy the L^∞ bound (i).

Next we note that, without loss of generality, we can assume that Ω is of the form

$$\big\{(x', x_n) = (x_1, \ldots, x_{n-1}, x_n)\colon x' \in]0, 1[^{n-1}, f(x') < x_n < 1\big\} \tag{3}$$

for some Lipschitz function f and some orthonormal system of coordinates $x_1, \ldots, x_n$. Indeed every connected Lipschitz set Ω can be written as union of finitely many Lipschitz domains $\Omega_1, \ldots, \Omega_N$ of the form (3) so that $\Omega_i \cap \Omega_{i+1} \neq \emptyset$.

Therefore, arguing by contradiction, we assume the existence of a Lipschitz open set Ω of the form (3) and a sequence of maps $u_j:\Omega\to\mathbb{R}^m$ with $\|\nabla u_j\|_{L^\infty(\Omega)}\leqslant M$ and $\|d_K(\nabla u_j)\|_{L^\infty(\Omega)}\leqslant M$ such that

$$\min\biggl(\int_\Omega d^p_{K_1}(\nabla u_j)\,\mathrm{d}x,\int_\Omega d^p_{K_2}(\nabla u_j)\,\mathrm{d}x\biggr)\geqslant j\int_\Omega d^p_K(\nabla u_j)\,\mathrm{d}x. \tag{4}$$

Let $L:=2\,\mathrm{Lip}(f)$. We denote by $Q_1=Q_1(0)$ the box $]-1,1[^{n-1}\times\,]-L,L[$ and by $Q_r(x)$ the boxes $x+rQ_1$. It is then easy to check that the following connectedness property holds:

(C) For any sequence of points $x_j\in\Omega$ and any sequence of positive numbers $r_j\leqslant\mathrm{diam}(\Omega)$, there exists a subsequence of the sets $U_j=\frac{1}{r_j}(\Omega-x_j)\cap Q_1$ converging in measure to a connected open set U such that
- $|U|\geqslant\gamma>0$ for some constant γ independent of r_j and x_j;
- $\forall\delta\in\,]0,\gamma[$ there exists a connected open set $\widetilde U$ with $|U\setminus\widetilde U|<\delta$ and $\widetilde U\subset U\cap U_j$ for j large enough.

By considering a suitable subsequence we may assume that ∇u_j generates a gradient Young measure ν_x. From (4) and the uniform Lipschitz bound we deduce that $\mathrm{supp}\,\nu_x\subset K$ a.e., hence by incompatibility $\mathrm{supp}\,\nu_x\subset K_2$ a.e., say. Therefore there exists $c>0$ such that

$$\int_\Omega d^p_{K_1}(\nabla u_j)\,\mathrm{d}x\geqslant c\quad\text{for all } j\quad\text{and}\quad\int_\Omega d^p_{K_2}(\nabla u_j)\,\mathrm{d}x\to 0\quad\text{as } j\to\infty. \tag{5}$$

Next, we define $S_j:=\{x\in\Omega:\ d_{K_1}(\nabla u_j)\leqslant d_{K_2}(\nabla u_j)\}$ and $f_j,g_j:\mathbb{R}^n\to\mathbb{R}$ by $f_j:=\chi_{S_j}d^p_{K_2}(\nabla u_j)$, $g_j:=\chi_{\Omega\setminus S_j}d^p_{K_1}(\nabla u_j)$. If $|S_j|=0$ for some $j>1$, then $d_K(\nabla u_j(x))=d_{K_2}(\nabla u_j(x))\leqslant d_{K_1}(\nabla u_j(x))$ a.e. in Ω, in contradiction with (4). Therefore we may assume that $|S_j|>0$. Using the definition of S_j we have $\int_\Omega f_j=\int_\Omega d^p_{K_2}(\nabla u_j)\,\mathrm{d}x-\int_{\Omega\setminus S_j}d^p_{K_2}(\nabla u_j)\,\mathrm{d}x=\int_\Omega d^p_{K_2}(\nabla u_j)\,\mathrm{d}x-\int_{\Omega\setminus S_j}d^p_K(\nabla u_j)\,\mathrm{d}x$, hence (4) implies that $\int_\Omega f_j\geqslant(j-1)\int_\Omega d^p_K(\nabla u_j)\,\mathrm{d}x$. On the other hand (5) implies $\int_\Omega f_j\to 0$, and consequently

$$\int_\Omega f_j+\int_\Omega g_j=\int_{S_j}d^p_{K_2}(\nabla u_j)\,\mathrm{d}x+\int_{\Omega\setminus S_j}d^p_{K_1}(\nabla u_j)\,\mathrm{d}x\geqslant\int_{S_j}d^p_{K_1}(\nabla u_j)\,\mathrm{d}x+\int_{\Omega\setminus S_j}d^p_{K_1}(\nabla u_j)\,\mathrm{d}x\geqslant c.$$

Therefore, by taking a subsequence if necessary, we may assume that

$$\int_\Omega(f_j-g_j)\leqslant-\frac{c}{2}. \tag{6}$$

Let us fix j for the moment. For a.e. $x\in S_j$, $\frac{1}{|Q_r(x)|}\int_{Q_r(x)}f_j\to f_j(x)$ and $\frac{1}{|Q_r(x)|}\int_{Q_r(x)}g_j\to 0$ as $r\to 0$, by Lebesgue's differentiation theorem. Hence $\frac{1}{|Q_r(x)|}\int_{Q_r(x)}(f_j-g_j)\to d^p_{K_2}(\nabla u_j(x))>0$ as $r\downarrow 0$, by the definition of S_j and since K_1 and K_2 are disjoint. On the other hand as $r\to\mathrm{diam}(\Omega)$ by (6) we have

$$\frac{1}{|Q_r(x)|}\int_{Q_r(x)}(f_j-g_j)\to\frac{1}{(\mathrm{diam}\,\Omega)^n}\int_\Omega(f_j-g_j)\leqslant-\frac{c}{2(\mathrm{diam}\,\Omega)^n}. \tag{7}$$

Since $r\mapsto\frac{1}{|Q_r(x)|}\int_{Q_r(x)}(f_j-g_j)$ is continuous, we deduce the existence of a radius $r(x)\in(0,\mathrm{diam}\,\Omega)$ for which $\int_{Q_{r(x)}(x)}f_j=\int_{Q_{r(x)}(x)}g_j$. The set of boxes $\{Q_{r(x)}(x):\ x\in S_j\}$ forms a cover for S_j, so by the Besicovitch covering theorem there are finitely many subfamilies of disjoint boxes $\mathcal{Q}_k\subset\{Q_{r(x)}(x):\ x\in S_j\}$ for $k=1,\dots,N$ such that $\bigcup_{k=1}^N\mathcal{Q}_k$ forms a cover for S_j (though usually stated for balls, the Besicovitch covering theorem holds for cubes as well, see for instance [7] Theorem 1.1, and hence even for our boxes, since the linear transformation $(x_1,\dots,x_{n-1},x_n)\mapsto(x_1,\dots,x_{n-1},x_n/L)$ maps them into cubes). Then $\sum_{k=1}^N\sum_{Q\in\mathcal{Q}_k}\int_Q f_j\geqslant\int_\Omega f_j$, so there exists k such that

$$\sum_{Q\in\mathcal{Q}_k}\int_Q f_j\geqslant\frac{1}{N}\int_\Omega f_j\geqslant\frac{1}{N}(j-1)\int_\Omega d^p_K(\nabla u_j)\,\mathrm{d}x\geqslant\frac{1}{N}(j-1)\sum_{Q\in\mathcal{Q}_k}\int_{Q\cap\Omega}d^p_K(\nabla u_j)\,\mathrm{d}x. \tag{8}$$

Therefore there exists a box $Q = Q_{r_j}(x_j) \in \mathcal{Q}_k$ such that

$$\int_{Q_{r_j}(x_j)} f_j = \int_{Q_{r_j}(x_j)} g_j \geqslant \frac{1}{N}(j-1) \int_{Q_{r_j}(x_j)\cap\Omega} d_K^p(\nabla u_j)\,\mathrm{d}x. \tag{9}$$

Let $U_j = \frac{1}{r_j}(\Omega - x_j) \cap Q_1$, $\Sigma_j = \frac{1}{r_j}(S_j - x_j) \cap Q_1$ and $v_j : U_j \to \mathbb{R}^m$ be defined as

$$v_j(x) = \frac{u_j(r_j(x - x_j)) - (u_j)_{x_j,r_j}}{r_j},$$

where $(u_j)_{x_j,r_j}$ denotes the average of u_j in $Q_{r_j}(x_j) \cap \Omega$. Then $\|\nabla v_j\|_{L^\infty(U_j)} \leqslant M$ and

$$\int_{\Sigma_j\cap U_j} d_{K_2}^p(\nabla v_j) = \int_{U_j\setminus\Sigma_j} d_{K_1}^p(\nabla v_j) \geqslant \frac{j-1}{N}\int_{U_j} d_K^p(\nabla v_j). \tag{10}$$

From (C), we can assume that a suitable subsequence of U_j converges to a connected open set U with $|U| \geqslant \gamma$. Let $\delta \in\,]0,\gamma[$, to be fixed later. By (C) there exists a connected open set $\widetilde{U} \subset U$ with $|U \setminus \widetilde{U}| < \delta$, and $\widetilde{U} \subset U_j$ for sufficiently large j. After taking a further subsequence we may assume that the sequence $\{\nabla v_j\}$ generates a gradient Young measure ν_x for $x \in \widetilde{U}$. In particular from (10) we deduce that $\operatorname{supp}\nu_x \subset K$ for almost every $x \in \widetilde{U}$, hence by incompatibility

$$\int_{\widetilde{U}} d_{K_1}^p(\nabla v_j)\,\mathrm{d}x \to 0 \quad \text{as} \quad j \to 0 \quad \text{or} \int_{\widetilde{U}} d_{K_2}^p(\nabla v_j)\,\mathrm{d}x \to 0 \quad \text{as } j \to 0. \tag{11}$$

By the Lipschitz bound and (10) we also have $|\int_{\Sigma_j\cap\widetilde{U}} d_{K_2}^p(\nabla v_j) - \int_{\widetilde{U}\setminus\Sigma_j} d_{K_1}^p(\nabla v_j)| \leqslant M^p|U \setminus \widetilde{U}| \leqslant M^p\delta$. In either case from (11) we deduce that for large enough $j \in \mathbb{N}$ $\int_{\Sigma_j\cap\widetilde{U}} d_{K_2}^p(\nabla v_j) \leqslant 2M^p\delta$ and $\int_{\widetilde{U}\setminus\Sigma_j} d_{K_1}^p(\nabla v_j) \leqslant 2M^p\delta$, and also $\int_{\widetilde{U}} d_K^p(\nabla v_j) \leqslant M^p\delta$. But then, from the definition of Σ_j we get $\int_{\widetilde{U}} d_{K_2}^p(\nabla v_j) \leqslant 3M^p\delta$ and $\int_{\widetilde{U}} d_{K_1}^p(\nabla v_j) \leqslant 3M^p\delta$ for sufficiently large j, which contradicts disjointness of K_1 and K_2 if δ is chosen sufficiently small. $\square$

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