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Numerical Analysis/Partial Differential Equations

A mixed formulation and exact controllability approach for the computation of the periodic solutions of the scalar wave equation. (I): Controllability problem formulation and related iterative solution

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Abstract

In this Note we discuss an exact controllability based method for the computation of the time-periodic solutions of a scalar wave equation with constant coefficients. We take advantage of an equivalent mixed formulation of the wave problem to derive a related controllability problem taking place in $(L^2(\Omega))^{d+1}$ (assuming that $\Omega \subset R^d$). Compared to previous work, where the controllability problem takes place in a subspace of $H^1(\Omega) \times L^2(\Omega)$, we can compute the periodic solutions by solving the novel controllability problem by a conjugate gradient algorithm operating in $(L^2(\Omega))^{d+1}$. The finite dimensional realization of the above algorithm does not require special preconditioning (as it is the case when the control space is contained in $H^1(\Omega) \times L^2(\Omega)$, requiring then the solution of discrete elliptic problems to achieve preconditioning). The results of numerical experiments validating this novel approach will be presented in a further Note. ***To cite this article: R. Glowinski, T. Rossi, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Sur le calcul des solutions périodiques en temps de l'équation des ondes scalaire via formulation mixte et exacte contrôlabilité. (I) : Formulation et résolution itérative du problème de contrôle. Dans cette Note, on étudie une méthode, basée sur la contrôlabilité exacte, pour le calcul des solutions périodiques en temps d'une équation des ondes scalaire à coefficients constants. On y prend avantage d'une formulation mixte équivalente du problème d'ondes pour se rammener à un problème de contrôlabilité posé dans $(L^2(\Omega))^{d+1}$ (on suppose que $\Omega \subset R^d$). Comparé à des travaux précédents, où le problème de contrôlabilité est posé dans un sous-espace de $H^1(\Omega) \times L^2(\Omega)$, on peut calculer les solutions périodiques en résolvant le nouveau problème de contrôlabilité par un algorithme de gradient conjugué opérant dans $(L^2(\Omega))^{d+1}$. L' analogue discret de l'algorithme ci-dessus ne demande pas de préconditionnement sophistiqué (comme c'est le cas quand l'espace de contrôle est contenu dans $H^1(\Omega) \times L^2(\Omega)$, exigeant alors la résolution de problèmes elliptiques discrets pour préconditionner). Les résultats d'essais numériques validant la nouvelle approche feront l'objet d'une note ultérieure. ***Pour citer cet article : R. Glowinski, T. Rossi, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

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1. Introduction. Problem formulations

In previous publications [7,3,4,8,9], the authors and collaborators have addressed the computation of the time-periodic solutions of various types of linear wave equations, the main goal being to avoid the many computational difficulties associated with the solution of the related Helmholtz equation encountered in the frequency domain approach; for example, when the wave length is small compared to some characteristic length. The methodology they employed to achieve these goal and computations can be summarized as follows: (i) view the problem as an exact controllability one, the control variables being the Cauchy data at $t = 0$; (ii) assume that one has efficient solution methods for solving the associated Cauchy problem; (iii) solve the exact controllability problem by a least squares/conjugate gradient algorithm operating in a well-chosen Hilbert space (in those previous works, and in the scalar case, it was always a subspace of $H^1(\Omega) \times L^2(\Omega)$). After an appropriate finite difference or finite element space approximation, the above approach is requiring, at every iteration of the conjugate gradient algorithm, the solution of a discrete elliptic problem. For large multi-dimensional wave problems, possibly involving differential operators with discontinuous coefficients, the above requirement weights heavily on the overall performance of the method, albeit keeping it very competitive compared to solution methods for the equivalent Helmholtz equation. The above controllability approach has been shown (see, e.g., [1]) leading to well-posed problems in appropriate functional spaces.

The main goal of the present work is to show that the above methodology still applies if one uses a mixed formulation of the wave problem, with the definite advantage that this time the conjugate gradient algorithm operates in 'pure' $L^2(\Omega)$ type spaces, avoiding thus the recourse to elliptic solvers based preconditioners. The price to pay for this simplification is that we shall have to use, for example, *Raviart–Thomas mixed finite element methods* (cf. [2,10]) for the space discretization. These mixed methods are always more complicated to implement than the usual ones based on Lagrange finite elements, particularly if the complications of the geometry make unstructured meshes and curved elements necessary. On the other hand, Raviart–Thomas elements having a lot in common with edge elements (à la Nédélec, for example), the present work is a good preparation (and investment) if one intends to look at the time-periodic solutions of the Maxwell equations, written as a first order system.

We consider thus the following prototypical *wave problem*

$$c^{-2}\psi_{tt} - \Delta\psi = 0 \quad \text{in } Q\ (= \Omega \times (0,T)), \tag{1}$$

$$\psi = g \quad \text{on } \sigma\ (= \gamma \times (0,T)), \qquad c^{-1}\psi_t + \partial\psi/\partial\mathbf{n} = 0 \quad \text{on } \Sigma_{\text{ext}}\ (= \Gamma_{\text{ext}} \times (0,T)), \tag{2}$$

$$\psi(0) = \psi(T), \qquad \psi_t(0) = \psi_t(T). \tag{3}$$

In (1)–(3): (i) $T > 0$ is the *time-period* and Ω is a bounded domain of R^d ($d \geqslant 1$); Ω surrounds an obstacle ω with boundary γ and it is externally bounded by an (artificial) boundary Γ_{ext}, with outward unit normal vector $\mathbf{n}$, on which the (Sommerfeld) *radiation condition* (the second boundary condition in (2)) is imposed (see Fig. 1); (ii) g is a given T-periodic function of t defined over σ; (iii) c (> 0) is the wave propagation speed; (iv) $\forall t \in [0,T]$, $\psi(t)$ denotes the function $x \mapsto \psi(x,t) : \overline{\Omega} \mapsto R$; (v) $\psi_t = \partial\psi/\partial t$, $\psi_{tt} = \partial^2\psi/\partial t^2$ and $\Delta = \sum_{i=1}^{d} \partial^2/\partial x_i^2$.

To obtain a mixed formulation of problem (1)–(3), we introduce the functions v and $\mathbf{p}$ defined by

$$v = \partial\psi/\partial t, \qquad \mathbf{p} = \nabla\psi. \tag{4}$$

The pair $\{v, \mathbf{p}\}$ verifies:

$$\begin{aligned} &c^{-2}\partial v/\partial t - \nabla\cdot\mathbf{p} = 0 \quad \text{in } Q, \qquad \partial\mathbf{p}/\partial t - \nabla v = 0 \quad \text{in } Q,\\ &v + c\mathbf{p}\cdot\mathbf{n} = 0 \quad \text{on } \Sigma_{\text{ext}}, \qquad v = \partial g/\partial t \quad \text{on } \sigma,\\ &v(0) = v(T), \qquad \mathbf{p}(0) = \mathbf{p}(T). \end{aligned} \tag{5}$$

The first order system (4), (5) has a *Maxwell equation* 'flavor', albeit simpler. A (mixed) *variational formulation* of this system is given by

$$\int_\Omega \big(c^{-2}\partial v/\partial t - \nabla\cdot\mathbf{p}\big) w\,\mathrm{d}x = 0, \quad \forall w \in L^2(\Omega),\ \text{a.e. on } (0,T), \tag{6}$$

$$\int_\Omega (\partial\mathbf{p}/\partial t\cdot\mathbf{q} + v\nabla\cdot\mathbf{q})\,\mathrm{d}x + c\int_{\Gamma_{\text{ext}}} \mathbf{p}\cdot\mathbf{n}\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}\Gamma = \int_\gamma \partial g/\partial t\,\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}\gamma, \quad \forall\mathbf{q} \in \mathbf{Q},\ \text{a.e. on } (0,T), \tag{7}$$

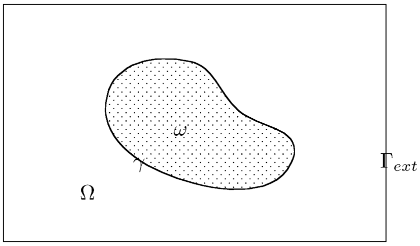

Fig. 1. Notations.

$$v(0) = v(T), \qquad \mathbf{p}(0) = \mathbf{p}(T), \tag{8}$$

with the space $\mathbf{Q}$ defined by $\mathbf{Q} = \{\mathbf{q} \mid \mathbf{q} \in H(\Omega; \mathrm{div}),\ \mathbf{q}\cdot\mathbf{n}|_{\Gamma_{\mathrm{ext}}} \in L^2(\Gamma_{\mathrm{ext}})\}$. We look for solutions $\{v, \mathbf{p}\}$ which belong at least to $L^2([0,T]; L^2(\Omega)\times\mathbf{Q}) \cap C^0([0,T]; L^2(\Omega)\times(L^2(\Omega))^d)$, a reasonable assumption if the function $\partial g/\partial t$ is smooth enough. Let us define the (*control*) space $\mathbf{E}$ by

$$\mathbf{E} = L^2(\Omega) \times \left(L^2(\Omega)\right)^d, \tag{9}$$

equipped with the following scalar product and corresponding norm:

$$(\mathbf{f}, \mathbf{g})_{\mathbf{E}} = \int_{\Omega} \left(c^{-2} f_0 g_0 + \mathbf{f}_1 \cdot \mathbf{g}_1\right) \mathrm{d}x, \quad \forall \mathbf{f} = \{f_0, \mathbf{f}_1\}, \mathbf{g} = \{g_0, \mathbf{g}_1\} \in \mathbf{E}. \tag{10}$$

An *exact controllability* problem (in $\mathbf{E}$), equivalent to (4), (5), reads as follows: Find $\mathbf{e} = \{e_0, \mathbf{e}_1\} \in \mathbf{E}$ such that

$$\begin{aligned} &c^{-2}\partial v/\partial t - \nabla\cdot\mathbf{p} = 0 \quad \text{in } Q, \qquad \partial\mathbf{p}/\partial t - \nabla v = 0 \quad \text{in } Q,\\ &v + c\mathbf{p}\cdot\mathbf{n} = 0 \quad \text{on } \Sigma_{\mathrm{ext}}, \qquad v = \partial g/\partial t \quad \text{on } \sigma,\\ &v(0) = e_0, \qquad \mathbf{p}(0) = \mathbf{e}_1, \end{aligned} \tag{11}$$

implies

$$v(T) = v(0), \qquad \mathbf{p}(T) = \mathbf{p}(0). \tag{12}$$

A *least-squares-conjugate gradient method* for the solution of problem (11), (12) will be considered in the following section.

2. Least-squares/conjugate gradient solution of the controllability problem (11), (12)

2.1. A least-squares formulation of problem (11), (12)

In order to solve problem (11), (12), we introduce the following *least-squares* formulation:

$$\text{Find } \mathbf{e} \in \mathbf{E} \text{ such that } J(\mathbf{e}) \leqslant J(\mathbf{f}), \quad \forall \mathbf{f} \in \mathbf{E}, \tag{13}$$

where, with $|\cdot| = \|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\| = \|\cdot\|_{(L^2(\Omega))^d}$,

$$J(\mathbf{f}) = 1/2\left[c^{-2}\left|v(T) - f_0\right|^2 + \left\|\mathbf{p}(T) - \mathbf{f}_1\right\|^2\right], \tag{14}$$

the functions v and $\mathbf{p}$ being obtained from $\mathbf{f} = \{f_0, \mathbf{f}_1\}$ via the solution of the following initial value problem:

$$\begin{aligned} &c^{-2}\partial v/\partial t - \nabla\cdot\mathbf{p} = 0 \quad \text{in } Q, \qquad \partial\mathbf{p}/\partial t - \nabla v = 0 \quad \text{in } Q,\\ &v + c\mathbf{p}\cdot\mathbf{n} = 0 \quad \text{on } \Sigma_{\mathrm{ext}}, \qquad v = \partial g/\partial t \quad \text{on } \sigma,\\ &v(0) = f_0, \qquad \mathbf{p}(0) = \mathbf{f}_1. \end{aligned} \tag{15}$$

In order to solve the least-squares problem (13) we will use a *conjugate gradient* algorithm operating in the Hilbert space $\mathbf{E}$ equipped with the scalar product defined by (10) and the corresponding norm. The implementation of such an algorithm is greatly facilitated by the knowledge of the *differential* $J'(\mathbf{f})$ of J at $\mathbf{f}$, $\forall\mathbf{f} \in \mathbf{E}$. This issue will be addressed in the following paragraph.

2.2. On the computation of $J'(\mathbf{f})$

Using methods discussed in, e.g., [7] we can show that the differential $J'(\mathbf{f})$ of J at $\mathbf{f}$, is given by

$$\big(J'(\mathbf{f}),\mathbf{g}\big)_{\mathbf{E}} = \int_\Omega \big[c^{-2}\big(f_0 - v(T) + v^*(0)\big)g_0 + \big(\mathbf{f}_1 - \mathbf{p}(T) + \mathbf{p}^*(0)\big)\cdot\mathbf{g}_1\big]\,\mathrm{d}x,$$
$$\forall \mathbf{f} = \{f_0, \mathbf{f}_1\},\ \mathbf{g} = \{g_0, \mathbf{g}_1\} \in \mathbf{E}, \tag{16}$$

or, equivalently, by

$$\big(J'(\mathbf{f}),\mathbf{g}\big)_{\mathbf{E}} = \int_\Omega [c^{-2}\big(v^*(0) - v^*(T)\big)g_0 + \big(\mathbf{p}^*(0) - \mathbf{p}^*(T)\big)\cdot\mathbf{g}_1]\,\mathrm{d}x, \quad \forall \mathbf{f} = \{f_0, \mathbf{f}_1\},\ \mathbf{g} = \{g_0, \mathbf{g}_1\} \in \mathbf{E}, \tag{17}$$

where the pair $\{v^*, \mathbf{p}^*\}$ is the unique solution of the following *adjoint* system:

$$\begin{aligned}
&c^{-2}\partial v^*/\partial t - \nabla\cdot\mathbf{p}^* = 0 \quad \text{in } \mathbf{Q}, \qquad \partial\mathbf{p}^*/\partial t - \nabla v^* = 0 \quad \text{in } \mathbf{Q},\\
&v^* - c\mathbf{p}^*\cdot\mathbf{n} = 0 \quad \text{on } \Sigma_{\text{ext}}, \qquad v^* = 0 \quad \text{on } \sigma,\\
&v^*(T) = v(T) - f_0, \qquad \mathbf{p}^*(T) = \mathbf{p}(T) - \mathbf{f}_1.
\end{aligned} \tag{18}$$

2.3. Conjugate gradient solution of the least squares problem (13)

Suppose that $\mathbf{e} = \{e_0, \mathbf{e}_1\}$ is the solution of the least squares problem (13); we have then:

$$\mathbf{e} \in \mathbf{E}, \quad \big(J'(\mathbf{e}),\mathbf{f}\big)_{\mathbf{E}} = 0, \quad \forall \mathbf{f} \in \mathbf{E}; \tag{19}$$

conversely, any solution of (19) solves problem (13). Since the operator J' is clearly affine continuous over $\mathbf{E}$, with its linear part positive semi-definite (at least), problem (19) is a *linear variational problem* in the Hilbert space $\mathbf{E}$. From its analogies with the linear variational problems whose conjugate gradient solution is discussed in, e.g., [5], Chapter 3, we are going to describe such an algorithm, operating in $\mathbf{E}$ for the solution of problem (13), (19). This algorithm reads as follows:

Initialization:

$$\mathbf{e}^0 = \big\{e_0^0, \mathbf{e}_1^0\big\} \quad \text{is given in } \mathbf{E}; \tag{20}$$

solve

$$\begin{aligned}
&c^{-2}\partial v^0/\partial t - \nabla\cdot\mathbf{p}^0 = 0 \quad \text{in } Q, \qquad \partial\mathbf{p}^0/\partial t - \nabla v^0 = 0 \quad \text{in } Q,\\
&v^0 + c\mathbf{p}^0\cdot\mathbf{n} = 0 \quad \text{on } \Sigma_{\text{ext}}, \qquad v^0 = \partial g/\partial t \quad \text{on } \sigma,\\
&v^0(0) = e_0^0, \qquad \mathbf{p}^0(0) = \mathbf{e}_1^0,
\end{aligned} \tag{21}$$

and then the adjoint system

$$\begin{aligned}
&c^{-2}\partial v^{*0}/\partial t - \nabla\cdot\mathbf{p}^{*0} = 0 \quad \text{in } Q, \qquad \partial\mathbf{p}^{*0}/\partial t - \nabla v^{*0} = 0 \quad \text{in } Q,\\
&v^{*0} - c\mathbf{p}^{*0}\cdot\mathbf{n} = 0 \quad \text{on } \Sigma_{\text{ext}}, \qquad v^{*0} = 0 \quad \text{on } \sigma,\\
&v^{*0}(T) = v^0(T) - e_0^0, \qquad \mathbf{p}^{*0}(T) = \mathbf{p}^0(T) - \mathbf{e}_1^0.
\end{aligned} \tag{22}$$

Next, define $\mathbf{g}^0 = \{g_0^0, \mathbf{g}_1^0\}$ and $\mathbf{w}^0 = \{w_0^0, \mathbf{w}_1^0\}$ by

$$g_0^0 = v^{*0}(0) - v^{*0}(T), \qquad \mathbf{g}_1^0 = \mathbf{p}^{*0}(0) - \mathbf{p}^{*0}(T), \tag{23}$$

and

$$\mathbf{w}^0 = \mathbf{g}^0, \tag{24}$$

respectively.

For $n \geqslant 0$, assuming that $\mathbf{u}^n, \mathbf{g}^n$ and $\mathbf{w}^n$ are known, the last two different from 0, we compute $\mathbf{u}^{n+1}, \mathbf{g}^{n+1}$, and if necessary, $\mathbf{w}^{n+1}$ as follows:

Descent:
solve

$$\begin{aligned}
&c^{-2}\partial\bar{v}^n/\partial t - \nabla\cdot\bar{\mathbf{p}}^n = 0 \quad \text{in } Q, \qquad \partial\bar{\mathbf{p}}^n/\partial t - \nabla\bar{v}^n = 0 \quad \text{in } Q,\\
&\bar{v}^n + c\bar{\mathbf{p}}^n\cdot\mathbf{n} = 0 \quad \text{on } \Sigma_{\text{ext}}, \qquad \bar{v}^n = 0 \quad \text{on } \sigma,\\
&\bar{v}^n(0) = w_0^n, \qquad \bar{\mathbf{p}}^n(0) = \mathbf{w}_1^n,
\end{aligned} \tag{25}$$

and then the adjoint system

$$\begin{aligned}
&c^{-2}\partial\bar{v}^{*^n}/\partial t - \nabla\cdot\bar{\mathbf{p}}^{*^n} = 0 \quad \text{in } Q, \qquad \partial\bar{\mathbf{p}}^{*^n}/\partial t - \nabla\bar{v}^{*^n} = 0 \quad \text{in } Q,\\
&\bar{v}^{*^n} - c\bar{\mathbf{p}}^{*^n}\cdot\mathbf{n} = 0 \quad \text{on } \Sigma_{\text{ext}}, \qquad \bar{v}^{*^n} = 0 \quad \text{on } \sigma,\\
&\bar{v}^{*^n}(T) = \bar{v}^n(T) - w_0^n, \qquad \bar{\mathbf{p}}^{*^n}(T) = \bar{\mathbf{p}}^n(T) - \mathbf{w}_1^n.
\end{aligned} \tag{26}$$

Define then $\bar{\mathbf{g}}^n = \{\bar{g}_0^n, \bar{\mathbf{g}}_1^n\}$ by

$$\bar{g}_0^n = \bar{v}^{*^n}(0) - \bar{v}^{*^n}(T), \qquad \bar{\mathbf{g}}_1^n = \bar{\mathbf{p}}^n(0) - \bar{\mathbf{p}}^n(T), \tag{27}$$

and compute

$$\rho_n = (\mathbf{g}^n, \mathbf{g}^n)_{\mathbf{E}} / (\bar{\mathbf{g}}^n, \mathbf{w}^n)_{\mathbf{E}}, \tag{28}$$

$$\mathbf{e}^{n+1} = \mathbf{e}^n - \rho_n\mathbf{w}^n, \tag{29}$$

$$\mathbf{g}^{n+1} = \mathbf{g}^n - \rho_n\bar{\mathbf{g}}^n. \tag{30}$$

Testing of convergence. Construction of the new descent direction:
If $(\mathbf{g}^{n+1}, \mathbf{g}^{n+1})_{\mathbf{E}}/(\mathbf{g}^0, \mathbf{g}^0)_{\mathbf{E}} \leqslant tol^2$ take $\mathbf{e} = \mathbf{e}^{n+1}$; else compute

$$\gamma_n = (\mathbf{g}^{n+1}, \mathbf{g}^{n+1})_{\mathbf{E}} / (\mathbf{g}^n, \mathbf{g}^n)_{\mathbf{E}}, \tag{31}$$

$$\mathbf{w}^{n+1} = g^{n+1} + \gamma_n\mathbf{w}^n. \tag{32}$$

Do $n = n + 1$ and return to (25).

3. Some remarks concerning the implementation of the controllability approach

Remark 1. The conjugate gradient algorithm (20)–(32) is particularly easy to implement, using for example the lowest order Raviart–Thomas mixed finite element approximation (and maybe also the one next to the lowest order). No (complicated) preconditioning is needed since we operate in $(L^2(\Omega))^{d+1}$. Actually, we can always use over E_h approximating E a scalar product, obtained by numerical integration, associated with a diagonal matrix.

Remark 2. The time-integration is particularly easy if one uses a staggered mesh. Suppose that $\Delta t = T/N$ and denote $n\Delta t$ by t^n and $(n + 1/2)\Delta t$ by $t^{n+1/2}$. We can for example discretize v over the set $\{t^n\}_{n=0}^N$ and $\mathbf{p}$ over the set $\{t^{n+1/2}\}_{n=0}^N$ (or the other way around) and use t^0 $(= 0)$ and t^N $(= T)$ to impose, respectively, the initial and final conditions on v, while (without loss of accuracy) the initial and final conditions on $\mathbf{p}$ will be imposed at $t^{1/2}$ $(= 1/2\Delta t)$ and $t^{N+1/2}$ $(= T + 1/2\Delta t)$, respectively. The coupled system can be easily discretized using centered second order accurate scheme which is 'almost' explicit (almost only since we will use a Crank–Nicolson type scheme to treat the radiation condition in the $\mathbf{p}$-equation). We can expect a stability condition such as $\Delta t \leqslant Ch$ (which is classical for this type of problems).

Remark 3. Mixed methods have been applied in, e.g., [6] to the numerical solution of exact boundary controllability problems for the wave equation, but to our knowledge, this is the first time they have been applied to the computation of time-periodic solutions by controllability methods.

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