Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 343 (2006) 653–656
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Dynamical Systems

On the conjugacy relation in ergodic theory

Matthew D. Foreman [a], Daniel J. Rudolph [b], Benjamin Weiss [c]

[a] *Mathematics Department, UC Irvine, Irvine, CA 92697, USA*
[b] *Mathematics Department, Colorado State University, Fort Collins, CO 80523, USA*
[c] *Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel*

Received 16 July 2006; accepted 5 September 2006

Presented by Gilles Pisier

Abstract

The set of pairs of transformations on the interval [0, 1] can be equipped with a standard Borel structure. We prove that the relation of conjugacy is not a Borel subset of this space, in fact it is complete analytic. Moreover, our construction proves that the two sets, $\{T\colon\ T$ is conjugate of $T^{-1}\}$, and $\{T\colon$ the centralizer of T is non-trivial$\}$ are complete analytic sets. ***To cite this article: M.D. Foreman et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Sur la relation de conjugation dans la théorie ergodique. L'ensemble des paires de transformations ergodiques de l'intervalle [0, 1] peut être muni d'une structure borélienne standard. Nous montrons que la relation de conjugaison n'est pas borélienne dans cet espace, en fait est analytique complète. Notre construction montre aussi que les ensembles $\{T\colon\ T$ est conjugué de $T^{-1}\}$ et $\{T\colon$ la centralisateur de T est non-trivial$\}$ sont des analytiques complets. ***Pour citer cet article : M.D. Foreman et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

1. Introduction

Paul Halmos concluded his classic lectures on ergodic theory [3] with a list of 'Unsolved Problems'. The third problem begins as follows:

"*The outstanding algebraic problem of ergodic theory is the problem of conjugacy: when are two transformations conjugate? This is vague, of course, but there are some interesting and quite specific yes-or-no questions connected with it that should be solved.*"

Halmos went on to specify three questions which were settled more than thirty five years ago. We will describe these briefly in the next section. Recently, Greg Hjorth in [4] answered several questions related to this problem motivated by the perspective of descriptive set theory. In particular, he showed that the equivalence relation of conjugate pairs

E-mail addresses: mforeman@math.uci.edu (M.D. Foreman), rudolphd@math.colostate.edu (D.J. Rudolph), weiss@math.huji.ac.il (B. Weiss).

1631-073X/$ – see front matter © 2006 Published by Elsevier Masson SAS on behalf of Académie des sciences.
doi:10.1016/j.crma.2006.09.011

of invertible measure preserving transformations is not a Borel set. His construction, for establishing this fact, relied in an essential way on non-ergodicity and left completely open the corresponding question concerning the conjugacy relation restricted to ergodic transformations. Our main goal in this Note is to outline a description of the complexity of this relation. Let us begin by giving a more precise description of the structures we are considering.

Fix one standard measure space $(X, \mathcal{B}, \mu)$, say the unit interval with Lebesgue measure, and denote by *MPT* the measure preserving invertible transformations on it. This becomes a Polish space when we take the topology induced from the strong operator topology on the unitary operators defined by elements from *MPT* on $L^2(X, \mathcal{B}, \mu)$. The ergodic transformations in *MPT*, denoted by *EMPT*, form a G_δ subset and thus have their own Polish topology giving rise to the same Borel structure. In the space of pairs (S, T), the conjugacy relation of pairs (S, T) for which there is some invertible measure preserving transformation R with $T = RSR^{-1}$ clearly form an analytic set since the set of triples (S, T, R) satisfying $T = RSR^{-1}$ is a closed subset in our topology. Hjorth showed that the conjugacy relation in $MPT \times MPT$ is not Borel and is in fact Borel equivalent to the most complicated kind of analytic set. We will do the same for the conjugacy relation in $EMPT \times EMPT$. Our results are based on the construction of a continuous mapping from the Polish space of countable trees $\mathcal{T}$ to the space of ergodic measure preserving transformations. In $\mathcal{T}$ the set of trees with at least one infinite branch is an analytic set and is even a complete analytic set. This means that any other analytic set can be 'reduced' to it by a Borel measurable mapping. The same is true for the set of trees with at least two infinite branches. Here is our main result:

Theorem 1.1. *There is a continuous map F from $\mathcal{T}$, the space of countable trees, to the ergodic measure preserving transformations with the following two properties*:

(i) *$\tau \in \mathcal{T}$ has an infinite branch if and only if $F(\tau)$ is conjugate to its inverse*;
(ii) *$\tau \in \mathcal{T}$ has at least two infinite branches if and only if $F(\tau)$ has a non-trivial centralizer, i.e. there is a measure preserving transformation S that commutes with T and is not a power of T.*

As an immediate corollary we will have:

Corollary 1.2. *The following sets are complete analytic sets, and in particular are not Borel*:

(i) $\{T \in EMPT\colon\ T \text{ is conjugate to } T^{-1}\}$;
(ii) $\{T \in EMPT\colon\ T \text{ has a non-trivial centralizer}\}$;
(iii) $\{(S, T) \in EMPT \times EMPT\colon\ S \text{ is conjugate to } T\}$.

2. Ergodic theory and descriptive set theory

To put our results in the proper perspective we review here briefly some earlier work on the conjugacy problem. The first major positive result was the classification of the ergodic transformation with pure point spectrum (pps) by von Neumann and Halmos. They showed that the group of eigenvalues is a complete invariant. This means that to decide whether or not two ergodic pps systems T and S are conjugate one merely has to compute the group of eigenvalues of each one of them and compare. From the point of view of descriptive set theory it is important to point out that this does not mean that the conjugacy relation restricted to the pps systems is simple. This is because all that one can actually do in a Borel measurable way is compute the eigenvalues of T as a sequence of complex numbers $\{e_n(T)\colon\ n \in \mathbb{N}\}$ and then T and S are conjugate if and only if the two sequences agree as unordered sets. This equivalence relation on sequences of distinct numbers is more complicated than the tail equivalence relation on sequences of zeros and ones which was used by J. Feldman [1] to show that there do not exist complete numerical Borel invariants which would characterize isomorphism of ergodic transformations. He used the Ornstein–Shields examples of non-Bernoulli K-automorphisms for this purpose, but in fact the pps systems already provide more complicated examples.

The great success of entropy, which is a numerical invariant that can be computed in a Borel measurable fashion and completely classifies the Bernoulli systems as shown by D. Ornstein [5], gave rise to some hope that more complicated invariants might be found which would classify all ergodic systems. The work of Feldman showed that nice invariants would not do (as we have already mentioned this could have been seen already in the pps). The group of eigenvalues is an example of just such a more complicated structure that is still countable. In [2] it was shown that the action of

MPT on EMPT is turbulent, in the sense of Hjorth, and as a consequence that countable structures cannot classify any collection of systems that is not of first category in EMPT. These results deal with the set EMPT itself, while our goal here is to study pairs of elements in EMPT and the nature of the equivalence relation itself.

We begin by fixing one standard measure space $(X, \mathcal{B}, \mu)$, say the unit interval with Lebesgue measure, and denote by *MPT* the measure preserving invertible transformations on it. This becomes a Polish space when we take the topology induced from the strong operator topology on the unitary operators defined by elements from *MPT* on $L^2(X, \mathcal{B}, \mu)$. In order to see that the ergodic transformations form a G_δ subset of the Polish space MPT fix a dense sequence of functions $\{g_j\}$ in L^2 and define the sets:

$$E(j, N, k) = \left\{ T \in MPT\colon \left\| \frac{1}{N} \sum_{n=1}^{N} T^n g_j - \int g_j \, \mathrm{d}\mu \right\| < 1/k \right\}.$$

Notice that these are open sets and that we can represent EMPT as the following:

$$EMPT = \bigcap_{(j,k)} \bigcup_{N} E(j, N, k).$$

As a G_δ set EMPT itself is a Polish space in the relative topology and we are concerned with this structure on EMPT.

Another way of getting the Borel structure we described above on *EMPT* may be derived from other universal models of *EMPT*. The easiest one to obtain is the following. Consider $\Omega = [0, 1]^{\mathbb{Z}}$ with the shift mapping σ. It is easy to see that among the invariant measures for this system every element T is represented. Indeed one can define the following mapping: $G(T) = \phi_T \circ \mu$ where ϕ_T is defined as follows:

$$\phi_T(X)_n = T^n(x).$$

This is just the mapping that sends each point of X to its orbit under T. It is easy to check that this mapping is continuous when the space of probability measures on Ω is endowed with the weak* topology, and that furthermore the system $(\Omega, G(T), \sigma)$ is conjugate to (X, μ, T).

One can also go the other way and map all the non-atomic shift invariant measures on Ω in a Borel fashion to *MPT*. This is done by uniformizing the proof of the fact that there is a Borel isomorphism (modulo an m-null set) between any non-atomic measure m on Ω and μ on X. Denoting such a mapping by H_m one maps a σ-invariant measure m to $T = H_m \sigma H_m^{-1}$ to obtain a mapping from the shift invariant measures to *MPT*. The standard Cantor–Bernstein argument now shows that the Borel structure on the ergodic shift invariant measures on Ω is the same as the one we initially described on *EMPT*.

In order to describe a countable tree we let $\mathbb{N}^*$ denote all finite words in the alphabet $\mathbb{N}$, including the empty word, and say that $u \in \mathbb{N}$ is a *predecessor* of v if it is an initial segment of v. A *tree* is a subset $\tau \subset \mathbb{N}^*$ that contains together with any word all of its predecessors. These words are called *nodes* of the tree and their *depth* is simply the length of the word. We say that the node u lies above v if it is a predecessor of it. Since $\mathbb{N}^*$ is a countable set we can consider the space of all trees $\mathcal{T}$, to be a subset of the Cantor set, and it is easily seen to be a closed subset. Thus the set of countably infinite trees, $\mathcal{T}$, as a G_δ subspace of a Polish space is Polish. The subset consisting of all those trees with an infinite path, which is simply a sequence of nodes u_i such that for all i, u_i lies above u_{i+1}, will be denoted by $\mathcal{T}_\infty$. The space of infinite paths for all trees can be identified with $\mathbb{N}^{\mathbb{N}}$ and it is easy to see that $\mathcal{T}_\infty$ is the projection of the closed subset in $\mathbb{N}^{\mathbb{N}} \times \mathcal{T}$ consisting of pairs $(f, T) \in \mathbb{N}^{\mathbb{N}} \times \mathcal{T}$ such that for all $n \in \mathbb{N}$, $f \restriction \{0, 1, \ldots, n\} \in T$. This shows that $\mathcal{T}_\infty$ is an analytic set.

It is a classical fact that not only is this set not Borel measurable but any analytic set $A \subset Y$ of a Polish space Y can be reduced to it in the sense that there is a Borel measurable mapping f from Y to $\mathcal{T}$ such that $A = f^{-1}(\mathcal{T}_\infty)$.

In the next section we will show how this set can be reduced to the conjugacy relation on ergodic maps.

3. The main construction

Sketch of proof. For the proof of Theorem 1.1 given a tree $\mathcal{T}$ we must construct an ergodic measure preserving transformation, $\mathcal{F}(\mathcal{T})$. We do so by constructing a shift invariant closed subset $\mathcal{Z}(\mathcal{T})$ of $\{1, 2, \ldots, a\}^{\mathbb{Z}}$ that has a unique invariant measure (which is then automatically ergodic). The shift invariant subset is constructed by describing in succession collections of finite blocks $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_n, \ldots$ in the basic alphabet $\{1, 2, \ldots, a\}$. The words in $\mathcal{W}_i$ all have the same length L_i, and the words in $\mathcal{W}_{i+1}$ are concatenations of words from $\mathcal{W}_i$. Furthermore, each $w \in \mathcal{W}_{i+1}$ contains each and every word from $\mathcal{W}_i$ the same number of times, say M_i so that $L_{i+1} = |\mathcal{W}_i| L_i M_i$. In addition we maintain the property that $w \in \mathcal{W}_{i+1}$ cannot be written in the form $u z_1 z_2 \cdots z_{L_{i+1}-1} v$ with the $z_i \in \mathcal{W}_i$, and u and v non-empty words. From this property it is easy to deduce the unique ergodicity of the closed subshift defined as the infinite words, all of whose finite blocks appear as subwords of some $w \in \mathcal{W}_i$ for some i. The tree $\mathcal{T}$ controls exactly how $\mathcal{W}_{i+1}$ is formed from $\mathcal{W}_i$ – but the parameters $|\mathcal{W}_i|$ and M_i are the same for all trees.

Paralleling the construction of the words $\mathcal{W}_n$, we construct groups of involutions G_i^n, $n \geqslant i$, with the G_i^n having a direct limit G_i as n tends to infinity. The group G_i is a direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$ with one generator associated to each node of $\mathcal{T}$ of depth i that has branches of arbitrary length in $\mathcal{T}$ passing through it. (Denote this collection of nodes $D_i = D_i(\mathcal{T})$.) There are also surjective homomorphisms $\rho_{i+1}^n : G_{i+1}^n \to G_i^n$ which converge to surjective homomorphism $\rho_{i+1} : G_{i+1} \to G_i$ which can be simply described as the map sending a node in D_{i+1} to its predecessor in D_i. The system (G_i, ρ_i) has an inverse limit denoted by G, and it is be non-trivial if and only if $\mathcal{T}$ has an infinite branch. From the way in which ρ_i is defined one can readily define uniquely a parity even/odd for element of G_i^n and G_i.

The connection between the groups G_i^n and the block structure of the words in $\mathcal{W}_n$ is the following. Each word w in $\mathcal{W}_n$ can be uniquely parsed into $\mathcal{W}_i$ words, and these are organized into clumps by an equivalence relation $\mathcal{Q}_i$. The elements of G_i^i act on these clumps freely, with the elements of odd parity reversing the order. The corresponding element in G_i^n acts on this clump structure of level i of w by either a diagonal action (for even parity) or a skew diagonal action (for odd parity), where the clumps in w are first reversed and then the diagonal action is applied.

As the construction proceeds, these clumps for increasing i have smaller and smaller diameter (with respect to $\bar{d}$, the average Hamming metric) so that an infinite branch through the tree leads to a conjugacy between the right shift on $\mathcal{Z}(\mathcal{T})$ and the left shift on $\mathcal{Z}(\mathcal{T})$ or, denoting this right shift by T, between T and T^{-1}.

The difficult part is to ensure that there is enough rigidity in the construction to guarantee that this is the only way in which T and T^{-1} can be conjugate. For this we keep track of the $\bar{d}$-distance between the clump structure at each level. The main device in going from $\mathcal{W}_n$ to $\mathcal{W}_{n+1}$ is a sufficiently random choice of the concatenations of the words from $\mathcal{W}_n$, while the action of a new element of G_i^{n+1} is defined by dividing the n-clumps into finer clumps and pairing between them. The basic technique to rule out a conjugacy $RTR^{-1} = T^{-1}$ is to use finite codes which approximate R extremely well in measure and then use the fact that finite codes must preserve rigid block structures when the block length is sufficiently long. These techniques are well established in ergodic theory ever since D. Ornstein's pioneering paper on the construction of a rank-1 mixing transformation.

Here is the basic idea. In passing from $\mathcal{W}_n$ to $\mathcal{W}_{n+1}$, we are 'randomly' concatenating the n-level clumps into which the words of $\mathcal{W}_n$ have been divided. These clumps are separated by a certain minimal distance in $\bar{d}$. A finite code defined on the shift space $\mathcal{Z}(\mathcal{T})$ whose length is much shorter than the length of the words in $\mathcal{W}_n$ preserves the identity patterns of the n-clumps in a word w on $\mathcal{W}_{n+1}$. Because of the minimal separation, and the fact distinct words in $\mathcal{W}_{n+1}$ are mutually random, this forces the image of w under the finite code to have essentially the same n-level clump structure and we can therefore identify the finite code with an element of the group G_n^n. This idea requires further elaboration to rule out large shifts. This is done by the randomness in the concatenations which guarantees that each individual word w in $\mathcal{W}_{n+1}$ looks random when viewed in parallel to a large shift of itself. To be sure there is much detail here that needs filling in, but these are the essential ideas. □

References

[1] J. Feldman, Borel structures and invariants for measurable transformations, Proc. Amer. Math. Soc. 46 (1974) 383–394.

[2] M. Foreman, B. Weiss, An anti-classification theorem for ergodic measure preserving transformations, J. Eur. Math. Soc. 6 (2004) 277–292.

[3] P.R. Halmos, Ergodic Theory, Chelsea Publishing Co, New York, NY, 1956.

[4] G. Hjorth, On invariants for measure preserving transformations, Fund. Math. 169 (2001) 51–84.

[5] D. Ornstein, Ergodic Theory, Randomness, and Dynamical Systems, Yale Mathematical Monographs, vol. 5, Yale University Press, New Haven, CT, London, 1974.