

Probability Theory

Hölder conditions for the local times of multiscale fractional Brownian motion

Raby Guerbaz

Department of Mathematics, Faculty of Sciences Semlalia, Cadi Ayyad University, 2390 Marrakesh, Morocco

Received 29 May 2006; accepted after revision 26 September 2006

Presented by Marc Yor

Abstract

We establish estimates for the local and uniform moduli of continuity of local times of multiscale fractional Brownian motion $\{X_\rho(t), t \geq 0\}$. We also give Chung's form of the law of the iterated logarithm for X_ρ , this proves that the results on local times are sharp up to multiplicative constant. **To cite this article:** R. Guerbaz, *C. R. Acad. Sci. Paris, Ser. I 343 (2006)*.

© 2006 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Résumé

Temps local du mouvement Brownien fractionnaire à multi-échelle. On étudie dans cette note les lois du logarithme itéré du temps local du mouvement Brownien fractionnaire à multi-échelle $\{X_\rho(t), t \geq 0\}$. On donne aussi la loi du logarithme itéré de type Chung pour X_ρ , ceci implique que les résultats concernant le temps local sont optimales. **Pour citer cet article :** R. Guerbaz, *C. R. Acad. Sci. Paris, Ser. I 343 (2006)*.

© 2006 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

The purpose of this Note is to establish sharp Hölder conditions for the local time of the multiscale fractional Brownian motion. This process is a generalization of the fractional Brownian motion (fBm) for which the Hurst parameter H is depending on the frequency. More precisely, for $K \in \mathbb{N}$, a (M_K) -multiscale fractional Brownian motion $(X(t), t \in (0, \infty))$ (M_K -fBm for brevity) is the Gaussian process with stationary increments defined by:

$$X_\rho(t) = \int_{\mathbb{R}} \frac{e^{it\lambda} - 1}{\rho(\lambda)} d\widehat{W}(\lambda),$$

where $dW(\lambda)$ is the random Brownian measure on $L^2(\mathbb{R})$ and for $i = 0, \dots, K$, there exist $(\omega_i, \sigma_i, H_i) \in (\mathbb{R}_+, \mathbb{R}_+,]0, 1[)$ such that $\rho(x) = \sigma_i^{-1} |x|^{H_i+1/2}$ for $|x| \in D_i = [\omega_i, \omega_{i+1}[$ with $\omega_0 = 0 < \omega_1 < \dots < \omega_K < \omega_{K+1} = +\infty$ by convention.

E-mail address: r.guerbaz@ucam.ac.ma (R. Guerbaz).

The M_K -fBm (called sometimes **Step fBm**, cf. [1]) was in particular introduced in order to relax the self-similarity property of fBm. The self-similarity is a form of invariance with respect to changes of time scale and it links the behavior at high frequencies to the behavior at low frequencies. This link restrict the use of the fBm as a model for real phenomenas, for instance biomechanics and finance. We refer to Bardet and Bertrand [2] and the references therein for details.

Let us recall some aspects of local time. We refer to the survey paper of Geman and Horowitz [4] and Xiao [9] for an excellent summary on local times of both random and nonrandom vector fields. Let $X = (X(t), t \in \mathbb{R}_+)$ be a real valued separable random process with Borel sample functions. The occupation measure of X is defined as follows:

$$\mu(A, B) = \lambda\{s \in A: X(s) \in B\} \quad \forall A \in \mathcal{B}(\mathbb{R}^+) \text{ and } \forall B \in \mathcal{B}(\mathbb{R}),$$

and λ is the Lebesgue measure on \mathbb{R}^+ . If $\mu(A, \cdot)$ is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} , we say that X has local times on A and define its local time, $L(A, \cdot)$, as the Radon–Nikodym derivative of $\mu(A, \cdot)$. In the notation $L(A, x)$, x is the so-called space variable and A is the time variable. Sometimes we write $L(t, x)$ instead of $L([0, t], x)$.

Recently, Boufoussi, Dozzi and Guerbaz [3] have studied local times and some related sample path properties for another generalization of the fBm called the multifractional Brownian motion, where the Hurst function $H(t)$ is a regular function of time instead of the frequency.

This Note is organized as follows. First, we establish Chung's form of the law of the iterated logarithm (LIL) for the M_K -fBm. We use this to establish a lower bound for the local moduli of continuity of the local time. We prove the upper bounds for the local, as well as uniform, moduli of continuity of the local times by using the concept of strong local nondeterminism (SLND for brevity) and an approach similar to that used in Xiao [8] and [9].

2. Main results

Our first result is an analogue of the famous Chung LIL for the M_K -fBm. We refer to the survey paper of Li and Shao [5] for the corresponding results for the Brownian motion and the fBm.

Theorem 1. *With probability one,*

$$\liminf_{\delta \rightarrow 0} \sup_{s \in [t, t+\delta]} \frac{|X_\rho(t) - X_\rho(s)|}{\delta^{H_K} / (\log |\log(\delta)|)^{H_K}} = \sigma_K C_{H_K}, \quad (1)$$

where C_{H_K} is the constant appearing in the Chung LIL of the fBm with Hurst exponent H_K in [6].

Proof. According to Monrad and Rootzén [6], the fBm B^{H_K} , with Hurst function H_K , satisfies (1). In addition, we have the following decomposition:

$$X_\rho(t) = \sigma_K B^{H_K}(t) + A(t),$$

where

$$A(t) = \sum_{j=0}^{K-1} \sigma_j \int_{D_j} \frac{e^{it\lambda} - 1}{|\lambda|^{H_j+1/2}} dW(\lambda) - \sigma_K \int_{\mathbb{R} \setminus D_K} \frac{e^{it\lambda} - 1}{|\lambda|^{H_K+1/2}} dW(\lambda).$$

Then, the theorem will be proved if we show that

$$\lim_{\delta \rightarrow 0} \sup_{s \in [t, t+\delta]} \frac{|A(t) - A(s)|}{\delta^{H_K} / (\log |\log(\delta)|)^{H_K}} = 0, \quad \text{a.s.}$$

According to Bardet and Bertrand ([2], Eq. (36)), $A(t)$ is a centered Gaussian process which satisfies:

$$E(A(t) - A(s))^2 \leq C|t - s|^2 \quad \text{for all } s, t \text{ close enough.}$$

Consequently, Theorem 2.1 of Monrad and Rootzén [6], implies that for any $H_K < \gamma < 1$,

$$\mathbb{P}\left(\sup_{s \in [t, t+\delta]} |A(t) - A(s)| \geq \varepsilon\right) \leq 1 - \exp(-C\delta\varepsilon^{-1/\gamma}).$$

Now, for $0 < \zeta < \gamma - H_K$, we consider $\delta_n = n^{2\gamma/(H_K+\zeta-\gamma)}$ and $\varepsilon_n = \delta_n^{H_K+\zeta}$ for all integer $n \geq 1$. Therefore

$$\sum_{n \geq 1} \mathbb{P}\left(\sup_{s \in [t, t+\delta_n]} |A(t) - A(s)| \geq \varepsilon_n\right) \leq \sum_{n \geq 1} (1 - \exp(-C\delta_n^{(\gamma-\zeta-H_K)/\gamma})) \approx C \sum_{n \geq 1} n^{-2} < +\infty.$$

It follows from the Borel Cantelli lemma that $\sup_{s \in [t, t+\delta_n]} |A(t) - A(s)| \leq \delta_n^{H_K+\zeta}$, almost surely for all n large enough. Moreover, for $\delta_{n+1} \leq \delta \leq \delta_n$, we have almost surely

$$\sup_{s \in [t, t+\delta]} |A(t) - A(s)| \leq \sup_{s \in [t, t+\delta_n]} |A(t) - A(s)| \leq \delta_n^{H_K+\zeta} \leq \delta^{H_K+\zeta} \left(\frac{\delta_n}{\delta_{n+1}}\right)^{H_K+\zeta} \leq 4^{\frac{\gamma(H_K+\zeta)}{\gamma-\zeta-H_K}} \times \delta^{H_K+\zeta}.$$

This completes the proof. \square

Remark 1. The main advantage of the previous proof is that it can be used to extend many LIL already known for the fBm to the M_K -fBm. For example, by the same arguments as above, we can prove the following LIL for the M_K -fBm which is analogue to ([5], Eq. 7.5), for fBm B^{H_K} :

$$\limsup_{\delta \rightarrow 0} \sup_{s \in [t, t+\delta]} \frac{|X_\rho(t) - X_\rho(s)|}{\delta^{H_K} (\log |\log(\delta)|)^{1/2}} = \sqrt{2}\sigma_K \left(\int_0^\infty \frac{1 - \cos(x)}{x^{2H_K+1}} dx \right)^{1/2} \text{ a.s.}$$

Our second main result establishes local and uniform modulus of continuity for the local time of X_ρ :

Theorem 2. *The M_K -fBm $\{X(t), t \geq 0\}$ admits a.s. a jointly continuous local time $L(t, x)$ satisfying for all $t \geq 0$:*

$$C_1^{-1} \leq \limsup_{\delta \rightarrow 0} \sup_{\mathbb{R}} \frac{L(t + \delta, x) - L(t, x)}{\delta^{1-H_K} (\log \log(\delta^{-1}))^{H_K}} \leq C_1 < \infty, \text{ a.s.} \tag{2}$$

and for any interval J , there exists a constant C_2 such that

$$\limsup_{\delta \rightarrow 0} \sup_{t \in J} \sup_{\mathbb{R}} \frac{L(t + \delta, x) - L(t, x)}{\delta^{1-H_K} (\log(1/\delta))^{H_K}} \leq C_2 < \infty \text{ a.s.}$$

To prove the upper bounds in Theorem 2, we need the following lemma:

Lemma 3. *The M_K -fBm is ϕ -strongly locally nondeterministic on any interval $(0, T]$, $T > 0$, with respect to the function $\phi(r) = r^{2H_K}$, i.e. for any $T > 0$ there exists a constant $C > 0$ such that for all $t \in (0, T]$ and all $0 < r \leq \min(|t|, 1)$, we have $V(X_\rho(t)/X_\rho(s), s \in (0, T], |t - s| \geq r) \geq C\phi(r)$.*

Proof. Working in the Hilbert space setting, the conditional variance is the square of the $L^2(P)$ -distance of $X_\rho(t)$ from the subspace generated by $\{X_\rho(s) : s \in (0, T], |t - s| \geq r\}$. We shall prove that there exists a constant $C > 0$ such that for every $t \in (0, T]$, $0 < r \leq \min\{|t|, 1\}$ the inequality $E(X_\rho(t) - \sum_{k=1}^m a_k X_\rho(s_k))^2 \geq C\phi(r)$ holds for all integers $m \geq 1$, all $a_k \in \mathbb{R}$, $k = 1, \dots, m$ and $s_k \in [0, T]$ satisfying $|t - s_k| \geq r$.

Since the intervals D_k are disjoint we have

$$E\left(X_\rho(t) - \sum_{k=1}^m a_k X_\rho(s_k)\right)^2 = \int_{\mathbb{R}} \left| e^{it\lambda} - 1 - \sum_{k=1}^m a_k (e^{is_k\lambda} - 1) \right|^2 \psi(\lambda) d\lambda$$

where we denote $\psi(\lambda) = \sum_{j=1}^K 1_{D_j}(|\lambda|) \times (\sigma_j^2/|\lambda|^{2H_j+1})$. Finally, since $\psi(\lambda) = \sigma_K^2/|\lambda|^{2H_K+1}$ for $|\lambda| > \omega_K$, then Proposition 8.4 of Pitt [7] implies the result. \square

Remark 2. In Section 8 of [7], the author has explained the way to obtain the local nondeterminism of Gaussian processes by comparing their spectral densities. This idea used in the previous proof suggests that the results of Pitt can be applied to prove the SLND.

Proof of Theorem 2. The two main ingredients needed to prove the upper bounds in the theorem are the strong local nondeterminism (Lemma 3) and the fact that

$$C_1|t-s|^{2H_K} \leq E(X_\rho(t) - X_\rho(s))^2 \leq C_2|t-s|^{2H_K} \quad \text{for } t, s \text{ sufficiently close.} \quad (3)$$

The proof follows now the lines of Theorems 1.1 and 1.2 in Xiao [8].

The lower bound in (2) is a consequence of the Chung LIL (1) and the following elementary computation:

$$\delta = \int_{\mathbb{R}} L([t, t + \delta], x) dx \leq 2 \sup_{x \in \mathbb{R}} L([t, t + \delta], x) \sup_{u \in [t, t + \delta]} |X_\rho(u) - X_\rho(t)|. \quad \square$$

Remark 3. The inequality (3) and the results of Section 28 in [4] imply that the local time of X_ρ has the same regularities, with respect to the space variable, as those given in ([4], p. 62, Table 2).

The following corollaries are immediate consequences of the strong local nondeterminism of the M_K -fBm (Lemma 3) and the Hölder continuity of the M_K -fBm. Their proofs rely on the two previous ingredients and similar arguments as in Xiao ([9], Section 3).

Corollary 4. *The following estimates on the small ball probability of M_K -fBm hold:*

(1) *Under the uniform norm:*

$$\exp(-K_1 \varepsilon^{-1/H_K}) \leq \mathbb{P}\left(\sup_{t \in [0,1]} |X_\rho(t)| \leq \varepsilon\right) \leq \exp(-K_2 \varepsilon^{-1/H_K});$$

(2) *Under the Hölder norm: For any $0 < \beta < H_K$ we have*

$$\exp(-K_1 \varepsilon^{-1/(H_K - \beta)}) \leq \mathbb{P}\left(\sup_{s, t \in [0,1]} \frac{|X_\rho(t) - X_\rho(s)|}{|t - s|^\beta} \leq \varepsilon\right) \leq \exp(-K_2 \varepsilon^{-1/(H_K - \beta)}).$$

Corollary 5. *The tail probability for the local time of M_K -fBm:*

$$-\log \mathbb{P}(L(1, 0) > x) \asymp x^{1/H_K} \quad \text{as } x \rightarrow \infty$$

where $f \asymp g$ denotes $0 < \liminf f(x)/g(x) < \limsup f(x)/g(x) < \infty$.

Acknowledgement

The author would like to thank the referee for several helpful corrections and comments that led to many improvements in the manuscript.

References

- [1] A. Benassi, P. Bertrand, S. Cohen, J. Iatas, Identification of the Hurst index of a step fractional Brownian motion, *Stat. Inference Stoch. Process.* 3 (2000) 101–111.
- [2] J.M. Bardet, P. Bertrand, Detecting abrupt change on the Hurst parameter of a multi-scale fractional Brownian motion with applications, in: *Int. Conf. on Self-Similarity and Applications*, Clermonts-Ferrand, France, 2002.
- [3] B. Boufoussi, M. Dozzi, R. Guerbaz, On the local time of the multifractional Brownian motion, *Stoch. Stoch. Rep.* 78 (2006) 33–49.
- [4] D. Geman, J. Horowitz, Occupation densities, *Ann. Probab.* 8 (1980) 1–67.
- [5] W. Li, Q.M. Shao, Gaussian processes: inequalities, small ball probabilities and applications, in: C.R. Rao, D. Shanbhag (Eds.), *Stochastic Processes: Theory and Methods*, in: *Handbook of Statistics*, vol. 19, 2001, pp. 533–597.
- [6] D. Monrad, H. Rootzén, Small values of Gaussian processes and functional laws of the iterated logarithm, *Probab. Theory Related Fields* 101 (1995) 173–192.
- [7] L.D. Pitt, Local times for Gaussian vector fields, *Indiana Univ. Math. J.* 27 (1978) 309–330.
- [8] Y. Xiao, Hölder conditions for the local times and the Hausdorff measure of the level sets of Gaussian random fields, *Probab. Theory Related Fields* 109 (1997) 129–157.
- [9] Y. Xiao, Strong local nondeterminism and the sample path properties of Gaussian random fields, Preprint, 2005.