

Partial Differential Equations

Localization of solutions for nonlinear elliptic problems with critical growth

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Abstract

We study the existence and the multiplicity of solutions for the problem $-\operatorname{div}(p(x)\nabla u) = u^{2^*-1} + \lambda u$, $u > 0$ in Ω and $u = 0$ on $\partial\Omega$, when the set of the minimizers for the weight p has multiple connected component. We study also the case where this set has one connected component and has complex topology. **To cite this article:** R. Hadiji et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).

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Résumé

Localisation des solutions pour un problème elliptique avec exposant critique de Sobolev. On étudie l'existence et la multiplicité de solutions du problème $-\operatorname{div}(p(x)\nabla u) = u^{2^*-1} + \lambda u$, $u > 0$ dans Ω et $u = 0$ sur $\partial\Omega$ dans le cas où l'ensemble de minima de p admet plusieurs composantes connexes. On s'intéresse également au cas où cet ensemble possède une seule composante connexe et une topologie complexe. **Pour citer cet article :** R. Hadiji et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).

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On considère le problème suivant :

$$(P_\lambda) \quad \begin{cases} -\operatorname{div}(p(x)\nabla u) = u^{2^*-1} + \lambda u, & u > 0 \text{ dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega \end{cases}$$

où Ω est un domaine borné de \mathbb{R}^n , $n \geq 3$, $2^* = \frac{2n}{n-2}$, $\lambda \in \mathbb{R}$ et $p: \bar{\Omega} \rightarrow \mathbb{R}$ une fonction positive dans $H^1(\Omega) \cap C(\bar{\Omega})$.

Soit $V(\Omega) = \{u \in H_0^1(\Omega): \int_\Omega |u(x)|^{2^*} dx = 1\}$, on définit

$$Q_\lambda(u) = \int_\Omega p(x)|\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx.$$

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On considère $S_\lambda(p) = \inf\{Q_\lambda(u) : u \in V(\Omega)\}$. Soit λ_1 la première valeur propre de l'opérateur $-\operatorname{div}(p(x)\nabla \cdot)$ dans $H_0^1(\Omega)$. On sait (voir [7]) qu'il existe $0 \leq \lambda^* < \lambda_1$ tels que pour $\lambda < \lambda^*$, $S_\lambda(p)$ n'est pas atteint et pour $\lambda > \lambda^*$, $S_\lambda(p)$ est atteint. On suppose qu'il existe $\bar{x} \in \Omega$ vérifiant

$$p(x) \leq \inf_{\Omega} p + c|x - \bar{x}|^\alpha \quad \forall x \in B(\bar{x}, r) \quad (1)$$

avec $c > 0$, $r > 0$ et $\alpha > 0$. Lorsque $n \geq 4$ et $\alpha > 2$ on a $\lambda^* = 0$, par contre si $n = 3$ ou $0 < \alpha \leq 2$ la situation est plus compliquée (voir [7]).

Dans cette Note, on s'intéresse au cas où $n \geq 4$ et $\alpha > 2$ (voir [8] pour les autres cas).

On utilise la fonction dite « barycentre » $\beta : V(\Omega) \rightarrow \mathbb{R}^n$ définie par

$$\beta(u) = \int_{\Omega} x |u(x)|^{2^*} dx \quad \forall u \in V(\Omega).$$

Nos résultats principaux sont les suivants :

Théorème 1. Soit Ω un domaine borné de \mathbb{R}^n , $n \geq 4$. On suppose qu'il existe $\bar{x} \in \Omega$ et un ouvert $A \subseteq \mathbb{R}^n$ tels que $\bar{x} \in A$ et si $\bar{\Omega} \cap \partial A \neq \emptyset$ on a $\inf_{\bar{\Omega} \cap \partial A} p > \inf_{\Omega} p$. Alors, il existe $\bar{\lambda}_A \in]0, \lambda_1]$ tel que, pour tout $\lambda \in]0, \bar{\lambda}_A[$, le problème (P_λ) admet au moins une solution $u_{\lambda,A}$ qui vérifie $\beta(\frac{u_{\lambda,A}}{\|u_{\lambda,A}\|_{L^{2^*}}}) \in A$ et $Q_\lambda(\frac{u_{\lambda,A}}{\|u_{\lambda,A}\|_{L^{2^*}}}) = S_\lambda(p, A)$ (voir (4)).

Corollaire 1. On suppose qu'il existe k ouverts disjoints dans \mathbb{R}^n , A_1, \dots, A_k , et k points $\bar{x}_1, \dots, \bar{x}_k$ dans Ω vérifiant les mêmes conditions que A et \bar{x} du Théorème 1. Alors, il existe $\bar{\lambda} \in]0, \lambda_1]$ tel que, pour tout $\lambda \in]0, \bar{\lambda}[$, le problème (P_λ) admet au moins k solutions distinctes $u_{\lambda,A_1}, \dots, u_{\lambda,A_k}$ vérifiant, pour $i = 1, \dots, k$, $\beta(\frac{u_{\lambda,A_i}}{\|u_{\lambda,A_i}\|_{L^{2^*}}}) \in A_i$ et $Q_\lambda(\frac{u_{\lambda,A_i}}{\|u_{\lambda,A_i}\|_{L^{2^*}}}) = S_\lambda(p, A_i)$ (voir (4)).

Théorème 2. Soit Ω un ouvert borné dans \mathbb{R}^n , $n \geq 4$. On suppose qu'il existe un ouvert A dans \mathbb{R}^n et un fermé $K \subseteq \Omega \cap A$ telles que, pour tout $\bar{x} \in K$, les conditions du Théorème 1 sont vérifiées pour des constantes convenables $c > 0$, $r > 0$ et $\alpha > 2$ (c, r, α sont indépendantes de \bar{x}). Alors, il existe $\bar{\lambda} > 0$ tel que, pour tout $\lambda \in]0, \bar{\lambda}[$, le problème (P_λ) admet au moins $\operatorname{cat}_A K$ solutions distinctes, où $\operatorname{cat}_A K$ représente le Ljusternik–Schnirelman catégorie de K dans A .

De plus, lorsque λ tend vers 0, chaque solution se concentre en un point minimum de p .

1. Introduction

Let us consider the following problem:

$$(P_\lambda) \quad \begin{cases} -\operatorname{div}(p(x)\nabla u) = u^{2^*-1} + \lambda u, & u > 0 \quad \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain of \mathbb{R}^n , $n \geq 3$, $2^* = \frac{2n}{n-2}$, $\lambda \in \mathbb{R}$ and $p : \bar{\Omega} \rightarrow \mathbb{R}$ is a positive function in $H^1(\Omega) \cap C(\bar{\Omega})$.

Let us denote by λ_1 the first eigenvalue of the operator $-\operatorname{div}(p(x)\nabla \cdot)$ in $H_0^1(\Omega)$.

Consider the set $V(\Omega) = \{u \in H_0^1(\Omega) : \int_{\Omega} |u(x)|^{2^*} dx = 1\}$ and define the functional

$$Q_\lambda(u) = \int_{\Omega} p(x) |\nabla u(x)|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx \quad \forall u \in V(\Omega).$$

For every $\lambda \in \mathbb{R}$, let us set

$$S_\lambda(p) = \inf\{Q_\lambda(u) : u \in V(\Omega)\}.$$

Notice that $S_\lambda(p)$ is a nonincreasing continuous function with respect to λ ; the definition of λ_1 implies $S_{\lambda_1}(p) = 0$; moreover, well-known concentration arguments show that $S_\lambda(p) = S \inf_{\Omega} p \quad \forall \lambda \leq 0$, where S denotes the best Sobolev constant, i.e. $S = \inf\{\int_{\Omega} |\nabla u|^2 dx : u \in V(\Omega)\}$.

The infimum $S_\lambda(p)$ is not achieved for any $\lambda < \lambda^*$, where $\lambda^* = \max\{\lambda \in \mathbb{R} : S_\lambda(p) = S \inf_{\Omega} p\}$; on the contrary, $S_\lambda(p)$ is achieved for every $\lambda > \lambda^*$ and, if $\lambda \in]\lambda^*, \lambda_1[$, the minimizing functions give rise to solutions of problem (P_λ) (taking into account the homogeneity of the nonlinear term).

The number λ^* depends on the behaviour of the function $p(x)$ near its minimum points (see [7,15]). In particular, if there exists $\bar{x} \in \Omega$ such that $p(x) \leq \inf_{\Omega} p + c|x - \bar{x}|^\alpha \forall x \in B(\bar{x}, r)$ for suitable positive constants c and r , then $\lambda^* = 0$ when $n \geq 4$ and $\alpha > 2$ (while the situation is more complex if $n = 3$ or $0 < \alpha \leq 2$).

2. Main results

In this Note, for the sake of simplicity, we consider only this case (see [8] for the other cases); for $\lambda > 0$ small enough, we obtain solutions which are localized near a prescribed region of minimizers for p ; as λ tends to 0, these solutions concentrate as Dirac masses near minimum points of $p(x)$ and, after rescaling, converge to minimizing functions for the best Sobolev constant (see [14]); moreover, we exploit this concentration phenomenon in order to relate the number of solutions to the geometrical properties of the set of the minimizers of p .

When $p = 1$ and $\lambda \rightarrow 0^+$ problem (P_λ) was studied in [13]. The effect of weights in front of the nonlinearity has been considered by several authors (see for example [1,9] and the references therein). Note that, in subcritical case, Musso and Passaseo obtained the existence and multiplicity of nontrivial solutions for problems having asymptotically singular weights in divergence form operators (see [11]).

We shall use the barycenter function $\beta : V(\Omega) \rightarrow \mathbb{R}^n$ defined by

$$\beta(u) = \int_{\Omega} x |u(x)|^2 dx \quad \forall u \in V(\Omega). \tag{2}$$

Lemma 1. Assume that $n \geq 4$ and that there exist $\bar{x} \in \Omega$ and $\alpha > 2$ such that

$$p(x) \leq \inf_{\Omega} p + c|x - \bar{x}|^\alpha \quad \forall x \in B(\bar{x}, r) \tag{3}$$

for suitable positive constants c and r . Moreover, assume that there exists an open set A in \mathbb{R}^n such that $\bar{x} \in A$ and, $\bar{\Omega} \cap \partial A \neq \emptyset$, $\inf_{\bar{\Omega} \cap \partial A} p > \inf_{\Omega} p$. Then, there exists $\tilde{\lambda}_A > 0$ such that the infimum

$$S_\lambda(p, A) = \inf\{Q_\lambda(u) : u \in V(\Omega), \beta(u) \in A\} \tag{4}$$

is achieved for every $\lambda \in]0, \tilde{\lambda}_A[$.

Proof. Notice that $\{u \in V(\Omega) : \beta(u) \in A\} \neq \emptyset$ (because $\bar{x} \in \Omega \cap A$) and $S_\lambda(p, A)$ is continuous and nonincreasing with respect to λ . Moreover (concentrating near \bar{x} the minimizing functions for the Sobolev constant and arguing as in [5,7,15]) one can show that $S_\lambda(p, A) = S \inf_{\Omega} p \forall \lambda \leq 0$ while (when $n \geq 4$ and $\alpha > 2$) $S_\lambda(p, A) < S \inf_{\Omega} p \forall \lambda > 0$.

Now, if $\{u \in V(\Omega) : \beta(u) \in \partial A\} \neq \emptyset$, let us consider the infimum

$$S_\lambda(p, \partial A) = \inf\{Q_\lambda(u) : u \in V(\Omega), \beta(u) \in \partial A\}.$$

It is clear that also $S_\lambda(p, \partial A)$ is continuous and nonincreasing with respect to λ .

Let us prove that $S_0(p, \partial A) > S \inf_{\Omega} p$. In fact, it is clear that $S_0(p, \partial A) \geq S \inf_{\Omega} p$. Arguing by contradiction, assume that the equality holds and consider a minimizing sequence $(u_i)_{i \in \mathbb{N}}$, i.e. $u_i \in V(\Omega)$, $\beta(u_i) \in \partial A \forall i \in \mathbb{N}$ and $\lim_{i \rightarrow +\infty} \int_{\Omega} p(x) |\nabla u_i(x)|^2 dx = S \inf_{\Omega} p$. Since $\int_{\Omega} p(x) |\nabla u(x)|^2 dx \geq \inf_{\Omega} p \int_{\Omega} |\nabla u(x)|^2 dx$, it follows that $\lim_{i \rightarrow +\infty} \int_{\Omega} |\nabla u_i(x)|^2 dx = S$. Therefore, there exists $x_0 \in \bar{\Omega}$ such that (up to a subsequence) $|\nabla u_i|^2 \rightarrow S \delta_{x_0}$ and $|u_i|^{2^*} \rightarrow \delta_{x_0}$, where δ_{x_0} denotes the Dirac mass in x_0 (see [10]).

If $\bar{\Omega} \cap \partial A = \emptyset$, we have a contradiction because $\beta(u_i) \in \partial A \forall i \in \mathbb{N}$ implies $x_0 \in \partial A$. If $\bar{\Omega} \cap \partial A \neq \emptyset$, we have $x_0 \in \bar{\Omega} \cap \partial A$ which (under our assumptions) implies $p(x_0) > \inf_{\Omega} p$. As a consequence, we obtain

$$\lim_{i \rightarrow +\infty} \int_{\Omega} p(x) |\nabla u_i(x)|^2 dx = S p(x_0) > S \inf_{\Omega} p \tag{5}$$

which also gives a contradiction.

Therefore, we must have $S_0(p, \partial A) > S \inf_{\Omega} p$ and, since $S_\lambda(p, \partial A)$ depends continuously on λ , there exists $\tilde{\lambda}_A > 0$ such that $S_\lambda(p, \partial A) > S \inf_{\Omega} p \forall \lambda \in]0, \tilde{\lambda}_A[$. Hence, we have

$$S_\lambda(p, A) < S \inf_{\Omega} p < S_\lambda(p, \partial A) \quad \forall \lambda \in]0, \tilde{\lambda}_A[. \tag{6}$$

Now, we can prove that the infimum $S_\lambda(p, A)$ is achieved for every $\lambda \in]0, \tilde{\lambda}_A[$. In fact, using a well-known result of Brezis and Lieb (see [4]) and arguing as in [5], from the inequality $S_\lambda(p, A) < S \inf_\Omega p$ one can infer that every minimizing sequence for $S_\lambda(p, A)$ is relatively compact in $H_0^1(\Omega)$ (for the strong H_0^1 topology). Thus, up to a subsequence, every minimizing sequence $(u_i)_{i \in \mathbb{N}}$ converges in $H_0^1(\Omega)$ to a function $\bar{u} \in V(\Omega)$ such that $Q_\lambda(\bar{u}) = S_\lambda(p, A)$ and $\beta(\bar{u}) \in \bar{A}$.

In order to complete the proof, we need only to observe that $\beta(\bar{u}) \in A$. Indeed, if $\{u \in V(\Omega) : \beta(u) \in \partial A\} = \emptyset$, this fact is obvious; in the other case, it follows from the inequality $S_\lambda(p, A) < S_\lambda(p, \partial A)$ (see (6)). \square

Theorem 1. *Let Ω be a bounded domain of \mathbb{R}^n , $n \geq 4$, assume that there exists $\bar{x} \in \Omega$ and an open set $A \subseteq \mathbb{R}^n$ satisfying the same conditions as in Lemma 1.*

Then, there exists $\bar{\lambda}_A \in]0, \lambda_1]$ such that, for every $\lambda \in]0, \bar{\lambda}_A[$, problem (P_λ) has at least one solution $u_{\lambda, A}$ such that $\beta(\frac{u_{\lambda, A}}{\|u_{\lambda, A}\|_{L^{2^}}}) \in A$ and $Q_\lambda(\frac{u_{\lambda, A}}{\|u_{\lambda, A}\|_{L^{2^*}}}) = S_\lambda(p, A)$ (see (4)).*

Proof. Let us set $\bar{\lambda}_A = \min\{\tilde{\lambda}_A, \lambda_1\}$ (see Lemma 1). Thus, we have $S_\lambda(p, A) > 0 \forall \lambda \in]0, \bar{\lambda}_A[$. Lemma 1 implies that there exist a minimizing function $\bar{u}_{\lambda, A}$ for the infimum $S_\lambda(p, A)$; clearly, we can assume $\bar{u}_{\lambda, A} \geq 0$ (otherwise we replace $\bar{u}_{\lambda, A}$ by $|\bar{u}_{\lambda, A}|$). Since $\{u \in V(\Omega) : \beta(u) \in A\}$ is an open subset of $V(\Omega)$, it follows that $\bar{u}_{\lambda, A}$ is a critical point for the functional Q_λ constrained on $V(\Omega)$. Hence, taking into account that $S_\lambda(p, A) > 0$, one can easily verify that $u_{\lambda, A} = [S_\lambda(p, A)]^{(n-2)/4} \bar{u}_{\lambda, A}$ solves problem (P_λ) and satisfies the desired properties. \square

The following corollary shows that Lemma 1 may be also used to obtain multiplicity of solutions when the set of the minimizers for $p(x)$ consists of several connected components.

Corollary 1. *Let Ω be a bounded domain of \mathbb{R}^n , $n \geq 4$, and assume that there exist k pairwise disjoint open sets in \mathbb{R}^n , A_1, \dots, A_k , and k points $\bar{x}_1, \dots, \bar{x}_k$ in Ω such that, for $i = 1, \dots, k$, \bar{x}_i and A_i satisfy the same conditions as \bar{x} and A in Lemma 1.*

Then, there exists $\bar{\lambda} \in]0, \lambda_1]$ such that, for every $\lambda \in]0, \bar{\lambda}[$, problem (P_λ) has at least k distinct solutions $u_{\lambda, A_1}, \dots, u_{\lambda, A_k}$ satisfying, for $i = 1, \dots, k$, $\beta(\frac{u_{\lambda, A_i}}{\|u_{\lambda, A_i}\|_{L^{2^}}}) \in A_i$ and $Q_\lambda(\frac{u_{\lambda, A_i}}{\|u_{\lambda, A_i}\|_{L^{2^*}}}) = S_\lambda(p, A_i)$ (see (4)).*

The proof is a direct consequence of Lemma 1 and Theorem 1.

Remark 1. When λ tends to 0, for each $i = 1, \dots, k$ the solution u_{λ, A_i} given by Corollary 1 tends to concentrate (up to a subsequence) as a Dirac mass near a point $x_i \in A_i$ such that $p(x_i) = \inf_\Omega p$, and, after rescaling, converges to a minimizing function for the best Sobolev constant. This property follows easily taking into account that $\lim_{\lambda \rightarrow 0} S_\lambda(p, A_i) = S \inf_\Omega p$; in fact, as a consequence, we have

$$\lim_{\lambda \rightarrow 0} \|u_{\lambda, A_i}\|_{L^{2^*}}^{-2} \int_\Omega p(x) |\nabla u_{\lambda, A_i}(x)|^2 dx = S \inf_\Omega p$$

which also implies

$$\lim_{\lambda \rightarrow 0} \|u_{\lambda, A_i}\|_{L^{2^*}}^{-2} \int_\Omega |\nabla u_{\lambda, A_i}(x)|^2 dx = S.$$

Corollary 1 guarantees the existence of multiple solutions of problem (P_λ) in a case where the set of the minimizers for the weight $p(x)$ has multiple connected components; however, notice that we can have multiple solutions even if this set has only one connected component but it has complex topology. For example, the following result holds:

Theorem 2. *Let Ω be a bounded domain of \mathbb{R}^n , $n \geq 4$, and assume that there exist an open set A in \mathbb{R}^n and a closed set $K \subseteq \Omega \cap A$ such that, for every $\bar{x} \in K$, the conditions required in Lemma 1 are satisfied for suitable constants $c > 0$, $r > 0$ and $\alpha > 2$ (c, r, α are independent of \bar{x}).*

Then, there exists $\bar{\lambda} > 0$ such that, for every $\lambda \in]0, \bar{\lambda}[$, problem (P_λ) has at least $\text{cat}_A K$ distinct solutions, where $\text{cat}_A K$ denotes the Ljusternik–Schnirelman category of K in A , i.e. the smallest integer h for which there exist h closed subsets of A , we call K_1, \dots, K_h , which are contractible in A and such that $K \subseteq \bigcup_{i=1}^h K_i$.

Moreover, as λ tends to 0, these solutions tend to concentrate as Dirac masses near points of A , which minimize the function $p(x)$, and, after rescaling, converge to minimizing functions for the best Sobolev constant.

Proof. Since $n \geq 4$ and $\alpha > 2$, we have $S_\lambda(p, A) < S \inf_\Omega p \ \forall \lambda > 0$ and if $\{u \in V(\Omega) : \beta(u) \in \partial A\} \neq \emptyset$, using Lemma 1 there exists $\bar{\lambda} > 0$ such that

$$\inf\{Q_\lambda(u) : u \in V(\Omega), \beta(u) \in \partial A\} > S \inf_\Omega p \ \forall \lambda < \bar{\lambda}. \tag{7}$$

The concentration arguments used in [3,5,7,15] show that, for every $\lambda > 0$, one can find a nonnegative radial function $\varphi_\lambda \in H_0^1(B(0, r))$ such that, if we set $\varphi_{\bar{x}, \lambda}(x) = \varphi_\lambda(x - \bar{x}) \ \forall \bar{x} \in K$ ($\varphi_{\bar{x}, \lambda}(x) = 0$ for $x \notin B(\bar{x}, r)$) and consider the set $\Phi_{K, \lambda} = \{\varphi_{\bar{x}, \lambda} : \bar{x} \in K\}$, then $\Phi_{K, \lambda} \subseteq V(\Omega)$ and

$$\sup\{Q_\lambda(u) : u \in \Phi_{K, \lambda}\} < S \inf_\Omega p. \tag{8}$$

Arguing as in [5,7,15] and taking into account (7) and (8), one can show that the sublevel

$$\left\{u \in V(\Omega) : \beta(u) \in A, Q_\lambda(u) \leq \sup_{\Phi_{K, \lambda}} Q_\lambda\right\} \tag{9}$$

is compact in $H_0^1(\Omega)$ for every $\lambda \in]0, \bar{\lambda}[$.

Therefore, if we denote by \tilde{k} the Ljusternik–Schnirelman category of $\Phi_{K, \lambda}$ in $\{u \in V(\Omega) : \beta(u) \in A, u \geq 0 \text{ in } \Omega\}$, general results of critical points theory guarantee that in this sublevel there exist at least \tilde{k} critical points for the functional Q_λ constrained on $V(\Omega)$, which give rise to \tilde{k} distinct solutions of (P_λ) .

Now, notice that $\tilde{k} \geq \text{cat}_A K$; in fact, if H_1, \dots, H_h are h closed subsets contractible in $\{u \in V(\Omega) : \beta(u) \in A, u \geq 0 \text{ in } \Omega\}$ and such that $\Phi_{K, \lambda} \subseteq \bigcup_{i=1}^h H_i$, then the sets K_1, \dots, K_h defined by $K_i = \{\bar{x} \in K : \varphi_{\bar{x}, \lambda} \in H_i\}$, for $i = 1, \dots, h$, are closed subsets of A , are contractible in A (as one can easily verify by using the barycenter function β), and $K = \bigcup_{i=1}^h K_i$.

Finally, let us remark that, as λ tends to 0, the asymptotic behaviour of the solutions follows easily taking into account that

$$\lim_{\lambda \rightarrow 0} S_\lambda(p, A) = \lim_{\lambda \rightarrow 0} \sup_{\Phi_{K, \lambda}} Q_\lambda = S \inf_\Omega p.$$

(arguing as in Remark 1). \square

Remark 2. The solutions obtained in Corollary 1 correspond to k local minimizers for the corresponding energy functional. Therefore, it is natural to expect that one could apply the mountain pass theorem (see [2]) to find other solutions, corresponding to higher critical values. But new difficulties arise, due to the presence of the weight $p(x)$; in fact, the presence of $p(x)$ may produce many energy levels where the Palais–Smale compactness condition is not satisfied; in particular, this condition may fail just at the mountain pass level.

However, under the assumptions required in Corollary 1, it is possible to find many higher energy solutions using a new approach (see [8]); in fact, for every subset $\{i_1, \dots, i_r\}$ of $\{1, \dots, k\}$, (by minimizing on a suitable manifold of codimension r) we construct a solution of (P_λ) which is the sum of r positive functions $u_{\lambda, i_1}, \dots, u_{\lambda, i_r}$ that, for $\lambda > 0$ small enough, are localized, respectively, near the prescribed regions A_{i_1}, \dots, A_{i_r} . Hence, for every $r = 1, \dots, k$, we obtain on the whole $\binom{k}{r}$ distinct r -peak solutions. Also in the case described in Theorem 2 it is natural to expect the existence of a higher energy solution when the set K is not contractible in A (i.e. $\text{cat}_A K \geq 2$). In this case (see [8]) the region where the weight $p(x)$ is large may play the same role as a hole in the domain (as in [6,12]); in general, the weight $p(x)$ may have the same effect as the domain shape on the existence and the multiplicity of solutions.

Finally, let us point out that the results presented in this Note may be also explained in terms of bifurcation with respect to the parameter λ . It is well known that a branch of positive solutions bifurcate from the first eigenvalue λ_1 ; some examples (see [8]) suggest that the existence of multiple solutions, given by Corollary 1 and Theorem 2, corresponds to secondary bifurcations from this branch of positive solutions.

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