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Partial Differential Equations

# On the large time behavior of solutions of fourth order parabolic equations and $\varepsilon$-entropy of their attractors

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**Abstract**

We study the large time behavior of solutions of a class of fourth order parabolic equations defined on unbounded domains. Specific examples of the equations we study are the *Swift–Hohenberg equation* and the *Extended Fisher–Kolmogorov equation*. We establish the existence of a global attractor in a local topology. Since the dynamics is infinite dimensional, we use the Kolmogorov $\varepsilon$-entropy as a measure, deriving a sharp upper and lower bound. ***To cite this article: M.A. Efendiev, L.A. Peletier, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***


**Résumé**

**Sur le comportement en temps grand des solutions d'équations paraboliques d'ordre quatre, et l'entropie de leurs attracteurs.** Nous étudions le comportement pour des grandes valeurs du temps des solutions d'une classe d'équations parabolique d'ordre quatre définie sur des domaines non bornés. Les examples spécifiques que nous considérons sont l'équation de *Swift–Hohenberg* et une généralisation de l'équation de *Fisher–Kolmogorov*. Nous démontrons l'existence d'un attracteur global dans une topologie locale, et nous obtenons des limites supérieure et inférieure de l'entropie de Kolmogorov. ***Pour citer cet article : M.A. Efendiev, L.A. Peletier, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***


## 1. Introduction

In this Note we give a description of the large time behavior of solutions of a family of well-known fourth order model equations of parabolic type of the form

$$u_t + \Delta^2 u + q\Delta u + f(u) = g \quad \text{in } \mathbf{R}^3, \tag{1.1}$$

where $q \in \mathbf{R}$ and $f(u)$ and $g = g(x)$ are given functions. Typical examples include the *Extended Fisher–Kolmogorov* equation (EFK) which arises in the study of bi-stable systems [1] and the *Swift–Hohenberg* equation (SH) which is

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used as a model equation in many studies of pattern formation [6]. Both equations can be brought into the form of Eq. (1.1) by suitable transformations of $t$, $x$ and $u$.

For one-dimensional bounded domains the large time behavior of solutions of Eq. (1.1) and pattern selection has been discussed in [4], in relation to the length of the domain.

In this Note we characterize the large time behavior in terms of the global attractor. For systems defined on unbounded domains the dynamics on the attractor (in contrast to the bounded domain) in most physically relevant cases is infinite dimensional [2], and classical tools like the (Hausdorff, fractal) dimension are no longer effective. Here the concept of Kolmogorov $\varepsilon$-entropy plays an important role. Recently this concept was applied to discuss dynamics described by second order equations in unbounded domains. In a series of papers in which the maximum principle was an important ingredient, the existence of the attractor was established and estimates of its $\varepsilon$-entropy were obtained [2,3]. However, for higher order equations such as (1.1) we encounter serious difficulties both due to the lack of maximum principle and lack of compactness, and here the interplay between different topologies will play a crucial role.

We prove the existence of a global attractor of solutions of (1.1) in a phase space endowed with a local topology, and present sharp estimates for the Kolmogorov $\varepsilon$-entropy. For (1.1) the local topology is appropriate because of the plethora of bounded stationary solutions such as periodic solutions, homoclinic and heteroclinic orbits and chaotic orbits, which were found for (EFK) and (SH) in one spatial dimension (cf. e.g. [5]). A strong topology would exclude such stationary solutions and lead to much simpler dynamics.

## 2. Preliminaries

We assume that the nonlinearity $f \in C^2(\mathbf{R})$ in (1.1) satisfies the following structure hypotheses:

$$\begin{cases} \text{H1:} \quad f(s)\cdot s \geqslant -c_1 + c_2|s|^{2+\delta} \quad \text{for } |s| \gg 1 \quad \text{and} \quad \left|f(s)\cdot s\right| \leqslant c_3|s| \quad \text{for } |s| \ll 1, \\ \text{H2:} \quad f'(s) \geqslant -c_4 \quad \text{for } s \in \mathbf{R}, \end{cases} \tag{2.1}$$

in which $c_1$, $c_2$, $c_3$, $c_4$ and $\delta$ are positive constants.

We shall be using function spaces which are typically designed for unbounded domains:

$$W_b^{\ell,p}\big(\mathbf{R}^3\big) = \Big\{u \in \mathcal{D}^1\big(\mathbf{R}^3\big):\ \|u\|_{b,\ell,p} \stackrel{\text{def}}{=} \sup_{x_0 \in \mathbf{R}^3} \big\|u, B^1_{x_0}\big\|_{\ell,p} < \infty\Big\}, \tag{2.2}$$

where $B^1_{x_0}$ is the unit ball centered at $x_0$ , as well as the weighted Sobolev spaces $L^p_\phi(\mathbf{R}^3)$ and $W^{\ell,p}_\phi(\mathbf{R}^3)$ (see [2,3]), with weight $\phi(x)$ such that $\phi(x) \to 0$ as $|x| \to \infty$. In particular we frequently use the space $\Phi_b \stackrel{\text{def}}{=} W_b^{4,2}(\mathbf{R}^3)$. In this Note we assume that

$$\text{H3:} \quad g \in L^2_b\big(\mathbf{R}^3\big). \tag{2.3}$$

## 3. Existence and uniqueness

The existence of a semigroup for the Cauchy Problem for (1.1) follows from the a priori estimate:

**Theorem 3.1.** *Let $u_0 \in \Phi_b$ and let $(f, g)$ in Eq.* (1.1) *satisfy* H1–H3. *Then Eq.* (1.1) *possesses a unique global solution $u(t)$ for $0 \leqslant t < \infty$, which satisfies the following estimate*:

$$\big\|u(t)\big\|_{\Phi_b} \leqslant Q\big(\big\|u(0)\big\|_{\Phi_b}\big)\mathrm{e}^{-\alpha t} + Q\big(\|g\|_{L^2_b}\big) := Q(u_0, g) \tag{3.1}$$

*in which $\alpha$ is a positive constant and $Q$ a generic function of its argument(s), which vanishes at the origin.*

The proof of Theorem 3.1 is based on successively multiplying Eq. (1.1) by $u(t)\phi_{\varepsilon,x_0}(x)$, $\partial_{x_i}(\phi_{\varepsilon,x_0}\partial_{x_i}u)$ and $\phi_{\varepsilon,x_0}\partial_t u$, integration by parts, and using assumptions H1–H3. This leads to an estimate for the $L^\infty$-norm of the nonlinearity. After that, the estimate (3.1) can be obtained exactly as for the linear case. Here $\phi_{\varepsilon,x_0}(x)$ is a smooth weight function such that $\phi_{\varepsilon,x_0}(x) \sim \mathrm{e}^{-\varepsilon|x-x_0|}$ as $|x| \to \infty$ where $x_0 \in \mathbf{R}^3$ and $\varepsilon > 0$.

We note that, for every two solutions $u_1(t)$ and $u_2(t)$ of (1.1), we have

$$\big\|u_1(t) - u_2(t)\big\|_{H^1_\phi} \leqslant C\mathrm{e}^{Kt}\big\|u_1(0) - u_2(0)\big\|_{L^2_\phi}, \tag{3.2}$$

where the constants $C$ and $K$ are independent of the choice of $u_1$ and $u_2$. Obviously, this estimate implies uniqueness of this solution.

The existence of a global (in time) solution of (1.1) is a standard consequence of (3.1). Therefore, Eq. (1.1) generates a semigroup $S_t : \Phi_b \to \Phi_b,\ S_t u_0 = u(t)$, and moreover $S_t : L^2_b(\mathbf{R}^3) \to W^{4,2}_b(\mathbf{R}^3),\ t > 0$.

## 4. Existence of a global attractor

**Definition 4.1.** A set $\mathcal{A} \subset \Phi_b$ is called a (locally compact) attractor of the semigroup $S_t$ if (1) $\mathcal{A}$ is bounded in $\Phi_b$ and compact in $\Phi_{\mathrm{loc}} := W^{4,2}_{\mathrm{loc}}(\mathbf{R}^3)$; (2) $S_t\mathcal{A} = \mathcal{A}$ for all $t > 0$; (3) $\mathcal{A}$ is an attracting set of a bounded set in the local topology of $\Phi_{\mathrm{loc}}$.

In the sequel we call an attractor as defined above a $(\Phi_b, \Phi_{\mathrm{loc}})$-attractor for the semigroup $S_t : \Phi_b \to \Phi_b$.

**Theorem 4.1.** *Let the nonlinearity $f$, and $g$ in Eq.* (1.1) *satisfy the hypotheses* H1–H3. *Then the semigroup $S_t$ defined above possesses a $(\Phi_b, \Phi_{\mathrm{loc}})$-attractor.*

The proof of Theorem 4.1 is based on the estimate (3.1), standard regularity theory and compactness of the embedding $W^{4+\delta,2}_b(\mathbf{R}^3) \subset\subset W^{4,2}_{\mathrm{loc}}(\mathbf{R}^3)$ for any $\delta > 0$.

**Remark.** We have proved that Eq. (1.1) has a $(\Phi_b, \Phi_{\mathrm{loc}})$-global attractor $\mathcal{A}$. When $g$ is constant it is easy to see that in general $\mathcal{A}$ cannot be compact in $\Phi_b$.

Indeed, if $\mathcal{A}$ is compact in $\Phi_b$, and $u_0 \in \mathcal{A}$, one can show that translation invariance implies that $u_0$ must be almost periodic. Since equilibria belong to $\mathcal{A}$, it follows that all equilibria must be almost periodic. It is well known however [5], that Eq. (1.1) for $q < 0$ and $f(u) = u^3 - u$ has kinks and pulse type stationary solutions. Plainly, they are not almost periodic.

## 5. Upper bound of the Kolmogorov entropy of the attractor

We prove the following upper bound for the Kolmogorov $\varepsilon$-entropy of $\mathcal{A}$:

**Theorem 5.1.** *Let the nonlinearity $f$, and $g$ in Eq.* (1.1) *satisfy the hypotheses* H1–H3. *Then*

$$\mathcal{H}_\varepsilon\big(\mathcal{A}|_{B^R_{x_0}}, W^{4,2}\big(B^R_{x_0}\big)\big) \leqslant C\left(R + \ln\frac{R_0}{\varepsilon}\right)^3 \ln\frac{R_0}{\varepsilon}, \tag{5.1}$$

*where the constants $C$ and $R_0$ do not depend on $\varepsilon$, $R$ and $x_0$.*

**Sketch of the proof.** Let $\phi = \phi_{R,x_0}(x)$ be the weight function introduced in Section 2, and let $\phi_{R,x_0}(x) \equiv 1$ in $B^R_{x_0}$. Then obviously

$$\mathcal{H}_\varepsilon\big(\mathcal{A}|_{B^R_{x_0}}, W^{4,2}\big(B^R_{x_0}\big)\big) \leqslant \mathcal{H}_\varepsilon\big(\mathcal{A}, W^{4,2}_\phi\big(\mathbf{R}^3\big)\big) \leqslant \mathcal{H}_{\varepsilon/L}\big(\mathcal{A}, L^2_\phi\big(\mathbf{R}^3\big)\big) \tag{5.2}$$

so that it suffices to prove (5.1) for $\mathcal{H}_{\varepsilon/L}(\mathcal{A}, L^2_\phi(\mathbf{R}^3))$. The latter one can deduce from the recurrence formula

$$\mathcal{H}_{\varepsilon/2}\big(\mathcal{A}, L^2_\phi\big(\mathbf{R}^3\big)\big) \leqslant C\left(R + \ln\frac{R_0}{\varepsilon}\right)^3 + \mathcal{H}_\varepsilon\big(\mathcal{A}, L^2_\phi\big(\mathbf{R}^3\big)\big), \tag{5.3}$$

which can be established following [3].

## 6. Lower bound of the Kolmogorov entropy of the attractor

We proceed in a more or less standard manner i.e., we aim at constructing the unstable manifold $\mathcal{M}^+(z_0)$ at some equilibrium point $z_0$. Because $\mathcal{M}^+(z_0) \subset \mathcal{A}$, we know that

$$\mathcal{H}_\varepsilon\big(\mathcal{A}|_{B^R_{x_0}}, W^{4,2}\big(B^R_{x_0}\big)\big) \geqslant \mathcal{H}_\varepsilon\big(\mathcal{M}^+(z_0), W^{4,2}\big(B^R_{x_0}\big)\big), \tag{6.1}$$

so that it suffices to compute the right-hand side in (6.1).

To this end, we assume that $g = 0$ and $f(0) = 0$, so that $u = 0$ is an equilibrium solution. Linearizing Eq. (1.1) about $u = 0$, we obtain the equation

$$w_t + \Delta^2 w + q\Delta w + f'(0)w = 0 \quad \text{in } \mathbf{R}^3. \tag{6.2}$$

We now assume that $u = 0$ is exponentially unstable, i.e.,

$$\text{H4:} \quad \sigma\big(-\Delta^2 - q\Delta - f'(0), L^2\big(\mathbf{R}^3\big)\big) \cap \{\operatorname{Re}\lambda > 0\} \neq \varnothing, \tag{6.3}$$

where $\sigma(L, W)$ denotes the spectrum of the linear operator $L$ in the space $W$. The characteristic equation corresponding to Eq. (6.2) has been studied in [4] (for the Swift–Hohenberg equation). From this analysis conditions on $q$ can readily be identified for which H4 is satisfied. We prove the following lower bound for the Kolmogorov entropy of $\mathcal{A}$, which in turn implies sharpness of the obtained estimates

**Theorem 6.1.** *Assume that $g = 0$ and $f(0) = 0$, and that* H1–H4 *are satisfied. Then, the entropy of the restrictions $\mathcal{A}|_{B_0^R}$ satisfies the following lower bound*:

$$\mathcal{H}_\varepsilon\big(\mathcal{A}|_{B_0^R}, L^\infty\big(B_0^R\big)\big) \geqslant C_* R^3 \ln \frac{\widetilde{R}_0}{\varepsilon}, \tag{6.4}$$

*where the positive constants $C_*$ and $\widetilde{R}_0$ are independent of $R$ and $\varepsilon$.*

**Remark.** We emphasize that in the case $g = \text{const}$, we have an additional structure on $\mathcal{A}$, namely we have an action of a $(1+3)$-parametric semigroup $\{\mathbb{S}_{(t,h)}, t \geqslant 0, h \in \mathbf{R}^3\}$ on $\mathcal{A}$, that is

$$\mathbb{S}_{(t,h)}\mathcal{A} = \mathcal{A}, \quad \mathbb{S}_{(t,h)} := S_t \circ T_h, \quad \mathbb{S}_{(t,h)}\colon \Phi_b \to \Phi_b \quad \text{with } T_h \circ S_t = S_t \circ T_h. \tag{6.5}$$

Here $\{T_h, h \in \mathbf{R}^3\}$ is a group of translations, $(T_h u)(x) := u(x+h)$. In a forthcoming paper, we will study dynamical properties of $\mathbb{S}_{(t,h)}$ in order to describe spatio-temporal complexity (for some values of $q$ in (1.1)) of the attractor.

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