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Number Theory

Lower bounds for the least common multiple of finite arithmetic progressions ☆

Shaofang Hong, Weiduan Feng

Mathematical College, Sichuan University, Chengdu 610064, P.R. China

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Abstract

Let u_0, r and n be positive integers such that $(u_0, r) = 1$. Let $u_k = u_0 + kr$ for $1 \leqslant k \leqslant n$. We prove that $L_n := \operatorname{lcm}\{u_0, u_1, \ldots, u_n\} \geqslant u_0(r+1)^n$ which confirms Farhi's conjecture (2005). Further we show that if $r < n$, then $L_n \geqslant u_0 r(r+1)^n$. ***To cite this article: S. Hong, W. Feng, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Minoration du plus petit commun multiple d'une progression arithmétique finie. Soit u_0, r et n des entiers positifs tels que $(u_0, r) = 1$, posons $u_k = u_0 + kr$ pour $1 \leqslant k \leqslant n$. Nous démontrons $L_n := ppcm(u_0, u_1, \ldots, u_n) \geqslant u_0(r+1)^n$, ce qui confirme la conjecture de Fahri (2005). De plus, nous montrons que si $r < n$ alors $L_n \geqslant u_0 r(r+1)^n$. ***Pour citer cet article : S. Hong, W. Feng, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

1. Introduction

Arithmetic progression is a basic subject in the study of Number Theory. The famous Dirichlet theorem (see, for instance, [1] or [5]) says that the arithmetic progression contains infinitely many primes if the first term and the common difference are coprime. Recently, Hong and Loewy [6] investigated the eigen structure of Smith matrices defined on a finite arithmetic progression and made some progress. Very recently, Green and Tao [3] have shown a significant theorem stating that the set of primes contains arbitrarily long arithmetic progression.

On the other hand, Hanson [4] and Nair [7] got the upper bound and lower bound of $\operatorname{lcm}\{1, \ldots, n\}$ respectively. Farhi [2] obtained some non-trivial lower bounds for the least common multiple of finite arithmetic progressions. Furthermore Farhi proposed the following conjecture:

Conjecture. *[2, Conjecture 2.5] Assume $u_0, r, n \in \mathbb{Z}^+$, $(u_0, r) = 1$ and $u_k = u_0 + kr$ for $1 \leqslant k \leqslant n$. Then $L_n := \operatorname{lcm}\{u_0, u_1, \ldots, u_n\} \geqslant u_0(r+1)^n$.*

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E-mail addresses: s-f.hong@tom.com, hongsf02@yahoo.com (S. Hong), d.d.feng@163.com (W. Feng).

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In this Note, we are interested in the least common multiple of finite arithmetic progressions. We exploit sharp lower bound for the least common multiple of a arithmetic progression with n terms. In particular, we show that the above conjecture is true. Under the condition $r < n$, we get an improved lower bound $L_n \geqslant u_0 r(r+1)^n$.

Throughout this Note, as usual, $[x]$ will denote the integer part of a given real number x. We say that a real number x is a multiple of a non-zero real number y if the quotient $\frac{x}{y}$ is an integer.

2. The main results

To show our main results, we first need a result of Farhi [2]. For the convenience to the readers, we here present an alternative proof using integrals. Throughout this section, we let $u_0, r, n \in \mathbb{Z}^+$ with $(u_0, r) = 1$, $u_k = u_0 + kr$ for $1 \leqslant k \leqslant n$ and $L_n = \mathrm{lcm}\{u_0, u_1, \ldots, u_n\}$.

Lemma 2.1. *For any positive integer n, L_n is a multiple of $\frac{u_0 u_1 \cdots u_n}{n!}$.*

Proof. We compute the integral $\int_0^1 x^{u_0/r-1}(1-x)^n \, \mathrm{d}x$ in two ways. First we use the binomial theorem to get that

$$\int_0^1 x^{u_0/r-1}(1-x)^n \, \mathrm{d}x = \int_0^1 x^{u_0/r-1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k \, \mathrm{d}x = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{r}{u_k}. \tag{1}$$

Second by using partial integral we have

$$\int_0^1 x^{u_0/r-1}(1-x)^n \, \mathrm{d}x = \frac{r}{u_0} \int_0^1 (1-x)^n \, \mathrm{d}x^{u_0/r} = \frac{n}{u_0/r} \int_0^1 x^{u_0/r}(1-x)^{n-1} \, \mathrm{d}x.$$

Continue to use partial integral for $n-1$ times, we get

$$\int_0^1 x^{u_0/r-1}(1-x)^n \, \mathrm{d}x = \frac{n!}{\frac{u_0}{r}(\frac{u_0}{r}+1)\cdots(\frac{u_0}{r}+n-1)} \int_0^1 x^{u_0/r+n-1} \, \mathrm{d}x = \frac{n! r^{n+1}}{u_0 u_1 \cdots u_n}. \tag{2}$$

So by (1) and (2) we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{u_k} = \frac{n! r^n}{u_0 u_1 \cdots u_n}. \tag{3}$$

By A denote the product of L_n and the left-hand side of (3). Clearly A is an integer. Multiplying both sides of (3) by L_n, we have $(n! r^n L_n)/(u_0 u_1 \cdots u_n) = A \in \mathbb{Z}$. So $L_n = (A/r^n)(u_0 u_1 \cdots u_n / n!)$. But $(r, u_0) = 1$ implies that $(r^n, u_0 u_1 \cdots u_n) = 1$. Thus $A_n := A/r^n$ is an integer. Then $L_n = A_n u_0 u_1 \cdots u_n / n!$ as required. This completes the proof of Lemma 2.1. □

Define $C_{n,k} := \frac{u_k \cdots u_n}{(n-k)!}$ for $0 \leqslant k \leqslant n$. Then we have the following lemma.

Lemma 2.2. *Let*

$$k_n := \max\left\{0, \left[\frac{n-u_0}{r+1}\right] + 1\right\}.$$

Then for any $0 \leqslant k \leqslant n$, we have $C_{n,k} \leqslant C_{n,k_n}$.

Proof. By the definition, we find the following relation

$$C_{n,k} = C_{n,k+1} \cdot \frac{u_k}{n-k} \tag{4}$$

for all $0 \leqslant k \leqslant n-1$. Let first $u_0 > n$. Then $k_n = 0$. Since $u_k > u_0$ and $n > n-k$, we have $\frac{u_k}{n-k} > 1$ for all $0 \leqslant k \leqslant n-1$. This implies immediately that $C_{n,0} > C_{n,1} > \cdots > C_{n,n}$. Thus Lemma 2.2 is true if $n < u_0$.

Now let $u_0 \leqslant n$. It is easy to see that $\frac{u_k}{n-k}$ is increasing as k increases. Note that $u_0/n \leqslant 1$, $u_{n-1}/(n-(n-1)) = u_{n-1} > 1$. Then there must be an integer l with $0 \leqslant l \leqslant n-2$ such that

$$\frac{u_l}{n-l} \leqslant 1 \quad \text{and} \quad \frac{u_{l+1}}{n-(l+1)} > 1. \tag{5}$$

So by (4) and (5) we obtain

$$C_{n,0} < \cdots < C_{n,l} \leqslant C_{n,l+1} > \cdots > C_{n,n}. \tag{6}$$

On the other hand, from (5) we derive that

$$\frac{n-u_0}{r+1} - 1 < l \leqslant \frac{n-u_0}{r+1} \;\Rightarrow\; l = \left[\frac{n-u_0}{r+1}\right].$$

Since $n \geqslant u_0, l \geqslant 0$. Thus $k_n = l+1$. Then by (6) we know that Lemma 2.2 holds if $n \geqslant u_0$. So Lemma 2.2 is proved. □

For any integer $0 \leqslant k \leqslant n$, define $L_{n,k} := \operatorname{lcm}\{u_k, \ldots, u_n\}$. Then $L_n = L_{n,0}$. By Lemma 2.1 we have

$$L_{n,k} = A_{n,k}\frac{u_k u_{k+1}\cdots u_n}{(n-k)!} = A_{n,k}C_{n,k} \tag{7}$$

with $A_{n,k} \in \mathbb{Z}^+$. It is obvious that L_n is a multiple of $L_{n,k}$ for all $0 \leqslant k \leqslant n$. Hence for all $0 \leqslant k \leqslant n$, by (7) we have $L_n \geqslant L_{n,k} \geqslant C_{n,k}$. Particularly $L_n \geqslant C_{n,k_n}$.

We can now prove the first main result:

Theorem 2.3. *Let $C_{n,k}$ and k_n be defined as above. Then $C_{n,k_n} \geqslant u_0(r+1)^n$. Consequently we have $L_n \geqslant u_0(r+1)^n$.*

Proof. We use induction on n to prove that $C_{n,k_n} \geqslant u_0(r+1)^n$. First if $n \leqslant u_0$, then by Lemma 2.2 we have

$$C_{n,k_n} \geqslant C_{n,0} = \frac{u_0u_1\cdots u_n}{n!} = u_0\frac{u_1}{1}\frac{u_2}{2}\cdots\frac{u_n}{n} = u_0(u_0+r)\left(\frac{u_0}{2}+r\right)\cdots\left(\frac{u_0}{n}+r\right) \geqslant u_0(r+1)^n.$$

Thus the conclusion is true for $n=1$ since $u_0 \geqslant 1$.

Assume that the claim holds for the case n. In what follows we prove that the claim is true for the case $n+1$. By the proof above, we may let $n > u_0$. Evidently we have $k_n \leqslant k_{n+1} \leqslant k_n + 1$. So we can divide the proof into the following two cases:

Case 1: $k_{n+1} = k_n$. Then we have

$$k_n = \left[\frac{n-u_0}{r+1}\right] + 1 = \left[\frac{n+1-u_0}{r+1}\right] + 1.$$

Hence we have

$$\frac{n+1-u_0}{r+1} < k_n. \tag{8}$$

Then

$$C_{n+1,k_{n+1}} = C_{n+1,k_n} = \frac{u_{k_n}\cdots u_n u_{n+1}}{(n+1-k_n)!} = C_{n,k_n}\cdot\frac{u_{n+1}}{n+1-k_n}, \tag{9}$$

By (8), we have

$$u_{n+1} - (r+1)(n+1-k_n) = u_0 + (n+1)r - (n+1)(r+1) + k_n(r+1) = u_0 - (n+1) + k_n(r+1) > 0.$$

So $\frac{u_{n+1}}{n+1-k_n} \geqslant r+1$. But the induction hypothesis tells us $C_{n,k_n} \geqslant u_0(r+1)^n$. Then by (9) we get $C_{n+1,k_{n+1}} \geqslant u_0(r+1)^{n+1}$ as required.

Case 2: $k_{n+1} = k_n + 1$. Then we have $k_n = k_{n+1} - 1 = [\frac{n+1-u_0}{r+1}]$. Thus

$$k_n \leqslant \frac{n+1-u_0}{r+1}. \tag{10}$$

So we have

$$C_{n+1,k_{n+1}} = C_{n+1,k_n+1} = \frac{u_{k_n+1}\cdots u_n u_{n+1}}{((n+1)-(k_n+1))!} = C_{n,k_n}\cdot\frac{u_{n+1}}{u_{k_n}}. \tag{11}$$

By (10) we have

$$\begin{aligned} u_{n+1}-(r+1)u_{k_n} &= u_0+(n+1)r-(r+1)(u_0+k_n r) = u_0+(n+1)r-(r+1)u_0-k_n r(r+1) \\ &\geqslant nr+r-u_0 r-r(n+1-u_0)=0. \end{aligned}$$

This implies that $\frac{u_{n+1}}{u_{k_n}} \geqslant r+1$. Then the desired result $C_{n+1,k_{n+1}} \geqslant u_0(r+1)^{n+1}$ follows immediately from (11) and the induction hypothesis. This completes the proof of the claim for case $n+1$. So Theorem 2.3 is proved. □

By Theorem 2.3, we know that Farhi's conjecture is true.

If we exploit the term $A_{n,k}$ in the identity (7), then we can improve the lower bound under certain condition as the following theorem shows.

Theorem 2.4. *Let $r < n$. Then we have $L_n \geqslant u_0 r(r+1)^n$.*

Proof. Letting $k = k_n$ in (7) gives us that

$$(n-k_n)!\cdot L_{n,k_n} = A_{n,k_n}\cdot u_{k_n}u_{k_n+1}\cdots u_n. \tag{12}$$

Suppose that $r \leqslant n-k_n$. Then $r \mid (n-k_n)!$. Since $(r,u_0)=1$, we have $(r, u_{k_n}u_{k_n+1}\cdots u_n)=1$. So from (12) we deduce that $r \mid A_{n,k_n}$. Hence $A_{n,k_n} \geqslant r$ and so $L_{n,k_n} = A_{n,k_n}C_{n,k_n} \geqslant u_0 r(r+1)^n$. Then the conclusion of Theorem 2.4 follows. Thus to prove Theorem 2.4, we need only to prove that $r \leqslant n-k_n$ which will be done in the following.

If $u_0 > n$, then $k_n = 0$ and $n-k_n = n > r$. If $u_0 = n$, then $k_n = 1$ and $n-k_n = n-1 \geqslant r$. If $u_0 < n$, then we consider the following three cases:

Case 1: $r < u_0 < n$. Then $k_n = [\frac{n-u_0}{r+1}]+1$. So we have $r+k_n \leqslant r+\frac{n-u_0}{r+1}+1 \leqslant \frac{(r+1)u_0+n-u_0}{r+1} < n$. Thus we have $r < n-k_n$ as required.

Case 2: $u_0 < r < 2r \leqslant n$. Then $r \geqslant 2$ and $n \geqslant 4$. Hence $k_n \leqslant \frac{n-u_0}{r+1}+1 \leqslant \frac{n-1}{3}+1 \leqslant \frac{n}{2}$. It follows that $r \leqslant n/2 \leqslant n-k_n$ as required.

Case 3: $u_0 < r < n < 2r$. Then $k_n \leqslant \frac{n-u_0}{r+1}+1 < \frac{2r-1}{r+1}+1 = 3-\frac{3}{r+1} < 3$. Since $k_n \geqslant 1$, k_n must be 1 or 2. If $k_n = 1$, then $r \leqslant n-1 = n-k_n$ as desired. If $k_n = 2$, then $r \neq n-1$. Otherwise we have $r = n-1$ which means

$$k_n = \left[\frac{n-u_0}{n-1+1}\right]+1 = 1.$$

This is impossible. So we have $r \leqslant n-2 = n-k_n$ as required.

The proof of Theorem 2.4 is complete. □

Remark. We point out that the conclusion of Theorem 2.4 may be false if the restricted condition $r < n$ does not hold. For example, let $u_0 = 1, r = n = 2$. Then $L_n = \text{lcm}\{1,3,5\} = 15$. But $u_0 r(r+1)^n = 18$. So we have $L_n < u_0 r(r+1)^n$.

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