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Numerical Analysis

An augmented discontinuous Galerkin method for elliptic problems ☆

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Abstract

In this Note we propose an augmented discontinuous Galerkin method for elliptic linear problems in the plane with mixed boundary conditions. Our approach introduces Galerkin least-squares terms, arising from constitutive and equilibrium equations, which allow us to look for the flux unknown in the local Raviart–Thomas space. The unique solvability is established avoiding the introduction of lifting operators and a Céa estimate is derived, which yields the rate of convergence of error, measured in an appropriate norm, being optimal respect to the h-version. We emphasize that for practical computations, this method reduces the degrees of freedom, with respect to the classical discontinuous Galerkin method. ***To cite this article: T.P. Barrios, R. Bustinza, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Une méthode de Galerkin discontinue augmentée pour des problèmes elliptiques. Dans cette Note, on se propose d'étudier une méthode de Galerkin discontinue augmentée pour des problèmes elliptiques bidimensionnels avec conditions mixtes à la frontière. L'approche introduit des termes de type Galerkin moindres carrés qui proviennent des équations constitutives et d'équilibre, et qui permettent de chercher les inconnues de flux dans des espaces Raviart–Thomas locaux. L'unicité des solutions est établie sans l'introduction d'opérateurs de relèvement. Une estimation de Céa est établie, qui montre que le taux de convergence de l'erreur, mesuré dans une norme appropriée, est optimal par rapport à la version h. Pour des expériences numériques, cette méthode présente l'avantage d'une réduction des degrés de liberté par rapport aux méthodes classiques de Galerkin discontinues. ***Pour citer cet article : T.P. Barrios, R. Bustinza, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

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Soit $\Omega \subseteq \mathbb{R}^2$ un ouvert borné et simplement connexe, de frontière polygonale Γ. Soient Γ_D et Γ_N deux parties de Γ telles que $|\Gamma_D| \neq 0$, $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$, et $\Gamma_D \cap \Gamma_N = \phi$.

On considère alors le problème suivant : étant données les fonctions $f \in L^2(\Omega)$, $g_D \in H^{1/2}(\Gamma_D)$, et $g_N \in L^2(\Gamma_N)$, trouver une fonction $u \in H^1(\Omega)$ solution de : $-\Delta u = f$ dans Ω, $u = g_D$ sur Γ_D, et $\frac{\partial u}{\partial \boldsymbol{\nu}} = g_N$ sur Γ_N, où $\boldsymbol{\nu}$ désigne la normale extérieure à Γ. En introduissant le gradient $\boldsymbol{\sigma} := \nabla u$ comme une inconnue additionnelle, le problème modèle peut être formulé comme : *trouver* $(\boldsymbol{\sigma}, u)$ *dans des espaces appropriés,* $\boldsymbol{\sigma} = \nabla u$ dans Ω, $-\operatorname{div} \boldsymbol{\sigma} = f$ dans Ω, $u = g_D$ sur Γ_D, et $\boldsymbol{\sigma} \cdot \boldsymbol{\nu} = g_N$ sur Γ_N. Dans le contexte des méthodes de Galerkin discontinues, ce problème est déja analysé dans [9], où les inconnues sont cherchées dans des espaces polynomiaux par morceaux qui appartiennent localement à H^1 et L^2. Dans ce travail, on se propose de développer une analyse d'erreur a priori pour une formulation augmentée DG, ce qui amène à chercher les inconnues dans des espaces appartenant localement à $H(\mathrm{div})$. Ceci s'obtient en ajoutant des termes de moindres carrés à la formulation mixte DG sous sa forme divergentielle, ce qui diffère des méthodes DG présentées dans [8] et [4]. De plus, cette approche suggère naturellement l'utilisation d'espaces Raviart–Thomas locaux pour l'approximation du vecteur des inconnues, ce qui permet une réduction des degrés de liberté par rapport à ceux requis dans [9].

1. Introduction

Let Ω be a bounded and simply connected domain in $\mathbb{R}^2$ with polygonal boundary Γ, and let Γ_D and Γ_N be parts of Γ such that $|\Gamma_D| \neq 0$, $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$, and $\Gamma_D \cap \Gamma_N = \phi$. Then, given $f \in L^2(\Omega)$, $g_D \in H^{1/2}(\Gamma_D)$, and $g_N \in L^2(\Gamma_N)$, we look for $u \in H^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in } \Omega, \quad u = g_D \quad \text{on } \Gamma_D, \quad \text{and} \quad \frac{\partial u}{\partial \boldsymbol{\nu}} = g_N \quad \text{on } \Gamma_N, \tag{1}$$

where $\boldsymbol{\nu}$ denotes the unit outward normal to Γ.

Now, as it is common in local discontinuous Galerkin method (cf. [1]) and, more generally, in mixed finite element methods, we introduce the gradient $\boldsymbol{\sigma} := \nabla u$ in Ω as additional unknown. In this way, (1) can be reformulated as the following problem in $\bar{\Omega}$: *Find* $(\boldsymbol{\sigma}, u)$ *in appropriate spaces such that*,

$$\boldsymbol{\sigma} = \nabla u \quad \text{in } \Omega, \quad -\operatorname{div} \boldsymbol{\sigma} = f \quad \text{in } \Omega, \quad u = g_D \quad \text{on } \Gamma_D, \quad \text{and} \quad \boldsymbol{\sigma} \cdot \boldsymbol{\nu} = g_N \quad \text{on } \Gamma_N. \tag{2}$$

We remark that problem (2) has already been analyzed in [9] using the local discontinuous Galerkin method, seeking the unknowns in piecewise polynomial spaces that locally belong to H^1 and L^2. In the present work, we will look for the vector unknown such that it locally belongs to $H(\mathrm{div})$, which motivates the employment of local Raviart–Thomas spaces to approximate it. This choice reduces the degrees of freedom than the needed in [5].

In order to attain this goal, we will augment the DG formulation obtained for (2) in divergence form, by adding suitable Galerkin least-squares terms, and then derive the optimal a priori error estimate, in the framework of the h-version.

The rest of the Note is organized as follows: in Section 2 we derive an augmented discontinuous Galerkin scheme, and prove its unique solvability. Finally, the optimal rates of convergence of our approximation is presented in Section 3.

2. The augmented DG formulation

In this section, we derive a discrete formulation for the linear model (1), applying the local discontinuous Galerkin method in divergence form, adding then suitable stabilization terms to obtain a well-posed formulation.

2.1. Meshes, averages and jumps

We let $\{\mathcal{T}_h\}_{h>0}$ be a family of shape-regular triangulations of $\bar{\Omega}$ (with possible hanging nodes) made up of straight-side triangles T with diameter h_T and unit outward normal to ∂T given by $\boldsymbol{\nu}_T$. As usual, the index h also denotes $h := \max_{T \in \mathcal{T}_h} h_T$. Then, given $\mathcal{T}_h$, its edges are defined as follows. An *interior edge* of $\mathcal{T}_h$ is the (nonempty) interior of $\partial T \cap \partial T'$, where T and T' are two adjacent elements of $\mathcal{T}_h$, not necessarily matching. Similarly, a *boundary edge* of $\mathcal{T}_h$ is the (nonempty) interior of $\partial T \cap \partial \Omega$, where T is a boundary element of $\mathcal{T}_h$. We denote by $\mathcal{E}_I$ the list of all interior edges of (counted only once) on Ω, by $\mathcal{E}_D$ and $\mathcal{E}_N$ the lists of all Dirichlet and Neumann boundary edges, respectively, and put $\mathcal{E} := \mathcal{E}_I \cup \mathcal{E}_D \cup \mathcal{E}_N$ the interior grid generated by the triangulation $\mathcal{T}_h$. Further, for each $e \in \mathcal{E}$, h_e represents its length. Also, in what follows we assume that $\mathcal{T}_h$ is of *bounded variation*, which means that there exists a constant $l > 1$, independent of the meshsize h, such that $l^{-1} \leqslant \frac{h_T}{h_{T'}} \leqslant l$ for each pair $T,\ T' \in \mathcal{T}_h$ sharing an interior edge.

Next, to define average and jump operators, we let T and T' be two adjacent elements of $\mathcal{T}_h$ and $[x]$ be an arbitrary point on the interior edge $e = \partial T \cap \partial T' \subset \mathcal{E}_I$. In addition, let v and $\boldsymbol{\tau}$ be scalar- and vector-valued functions, respectively, that are smooth inside each element $T \in \mathcal{T}_h$. We denote by $(v_{T,e}, \boldsymbol{\tau}_{T,e})$ the restriction of $(v_T, \boldsymbol{\tau}_T)$ to e. Then, we define the averages at $[x] \in e$ by:

$$\{v\} := \frac{1}{2}(w_{T,e} + w_{T',e}), \qquad \{\boldsymbol{\tau}\} := \frac{1}{2}(\boldsymbol{\tau}_{T,e} + \boldsymbol{\tau}_{T',e}).$$

Similarly, the jumps at $[x] \in e$ are given by

$$[\![v]\!] := v_{T,e}\boldsymbol{\nu}_T + v_{T',e}\boldsymbol{\nu}_{T'}, \qquad [\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}_{T,e} \cdot \boldsymbol{\nu}_T + \boldsymbol{\tau}_{T',e} \cdot \boldsymbol{\nu}_{T'}.$$

On boundary edges e, we set $\{v\} := v$, $\{\boldsymbol{\tau}\} := \boldsymbol{\tau}$, as well as $[\![v]\!] := v\boldsymbol{\nu}$ and $[\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau} \cdot \boldsymbol{\nu}$. Hereafter, div_h and ∇_h denote the piecewise divergence and gradient operators.

2.2. The augmented discrete formulation

Given a mesh $\mathcal{T}_h$, we proceed as in [9] (or [5]) and multiply each one of the equations (introduced at the introduction) by suitable test functions. We wish to approximate the exact solution $(\boldsymbol{\sigma}, u)$ of (2) by discrete functions $(\boldsymbol{\sigma}_h, u_h)$ in appropriate finite element space $\boldsymbol{\Sigma}_h \times \boldsymbol{\mathcal{V}}_h$ such that for all $T \in \mathcal{T}_h$ we have

$$\begin{aligned} &\int_T \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} + \int_T u_h \,\mathrm{div}\, \boldsymbol{\tau} - \int_{\partial T} \hat{u} \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h, \\ &\int_T \boldsymbol{\sigma}_h \cdot \nabla v - \int_{\partial T} v \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}_T = \int_T f v \quad \forall v \in V_h, \end{aligned} \tag{3}$$

where the *numerical fluxes* $\hat{u}$ and $\hat{\boldsymbol{\sigma}}$, which usually depend on u_h, $\boldsymbol{\sigma}_h$, and the boundary data, are chosen so that some compatibility conditions are satisfied (see [1]).

We are now ready to complete the DG formulation (3). Indeed, using the approach from [9] and [6], we define the numerical fluxes $\hat{u}$ and $\hat{\boldsymbol{\sigma}}$ for each $T \in \mathcal{T}_h$ as follows:

$$\hat{u}_{T,e} := \begin{cases} \{u_h\} - \boldsymbol{\beta} \cdot [\![u_h]\!] & \text{if } e \in \mathcal{E}_I, \\ g_D & \text{if } e \in \mathcal{E}_D, \\ u_h & \text{if } e \in \mathcal{E}_N, \end{cases} \quad \text{and} \quad \hat{\boldsymbol{\sigma}}_{T,e} := \begin{cases} \{\boldsymbol{\sigma}_h\} + \boldsymbol{\beta}[\![\boldsymbol{\sigma}_h]\!] - \alpha[\![u_h]\!] & \text{if } e \in \mathcal{E}_I, \\ \boldsymbol{\sigma}_h - \alpha(u_h - g_D)\boldsymbol{\nu} & \text{if } e \in \mathcal{E}_D, \\ g_N \boldsymbol{\nu} & \text{if } e \in \mathcal{E}_N, \end{cases} \tag{4}$$

where the auxiliary functions α (scalar) and $\boldsymbol{\beta}$ (vector), to be chosen appropriately, are single valued on each edge $e \in \mathcal{E}$ and such that they allow us to prove the optimal rates of convergence of our approximation. To this aim, we set $\alpha := \hat{\alpha}/\mathtt{h}$, and $\boldsymbol{\beta}$ as an arbitrary vector in $\mathbb{R}^2$. Hereafter, $\hat{\alpha} > 0$ is arbitrary, while $\mathtt{h}$ is defined by $\mathtt{h} := \max\{h_T, h_{T'}\}$, if $e \in \mathcal{E}_I$; and $\mathtt{h} := h_T$, if $e \in \mathcal{E}_D$. Moreover, and for notation purposes, we will consider from here on that $\boldsymbol{\beta} = (0,0)$ on $\mathcal{E}_N$.

Then, integrating by parts in the second equation in (3), summing up over all $T \in \mathcal{T}_h$, we arrive to: *Find* $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\mathcal{V}}_h$ *such that*

$$\begin{aligned} &\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} + \int_\Omega u_h \,\mathrm{div}_h\, \boldsymbol{\tau} - \int_{\mathcal{E}_I \cup \mathcal{E}_N} [\![\boldsymbol{\tau}]\!]\big(\{u_h\} - \boldsymbol{\beta} \cdot [\![u_h]\!]\big) = \int_{\mathcal{E}_D} g_D \boldsymbol{\tau} \cdot \boldsymbol{\nu}, \\ &-\int_\Omega v \,\mathrm{div}_h\, \boldsymbol{\sigma}_h + \int_{\mathcal{E}_I \cup \mathcal{E}_N} [\![\boldsymbol{\sigma}_h]\!]\big(\{v\} - \boldsymbol{\beta} \cdot [\![v]\!]\big) + \boldsymbol{\alpha}(u_h, v) = \int_\Omega f v + \int_{\mathcal{E}_D} \alpha g_D v + \int_{\mathcal{E}_N} g_N v, \end{aligned} \tag{5}$$

for all $(\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma}_h \times \mathcal{V}_h$, with $\boldsymbol{\alpha} : H^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h) \to \mathbb{R}$ being the bilinear form defined by:

$$\boldsymbol{\alpha}(w, v) := \int\limits_{\mathcal{E}_I \cup \mathcal{E}_D} \alpha [\![w]\!] \cdot [\![v]\!], \quad \forall w, v \in H^1(\mathcal{T}_h).$$

We point out that considering the identity

$$\int\limits_{\Omega} v \operatorname{div}_h \boldsymbol{\tau} - \int\limits_{\mathcal{E}_I} [\![\boldsymbol{\tau}]\!] \cdot \big(\{v\} - \boldsymbol{\beta} \cdot [\![v]\!]\big) - \int\limits_{\mathcal{E}_N} v \boldsymbol{\tau} \cdot \boldsymbol{\nu} = - \int\limits_{\Omega} \nabla_h v \cdot \boldsymbol{\tau}$$
$$+ \int\limits_{\mathcal{E}_I} \big(\{\boldsymbol{\tau}\} + \boldsymbol{\beta} [\![\boldsymbol{\tau}]\!]\big) \cdot [\![v]\!] + \int\limits_{\mathcal{E}_D} v \boldsymbol{\tau} \cdot \boldsymbol{\nu} \quad \forall (\boldsymbol{\tau}, v) \in \big[H^1(\mathcal{T}_h)\big]^2 \times H^1(\mathcal{T}_h), \tag{6}$$

which is obtained integrating by parts, it is easy to deduce an equivalent formulation to (5), whose unique solvability is analyzed in [5]. Now, since u really lives in the space $H^1(\Omega)$, $\boldsymbol{\sigma}$ belongs to $H(\operatorname{div}; \Omega)$, and our discrete unknowns are discontinuous, we approximate them by elements in $H^1(\mathcal{T}_h)$ and $H(\operatorname{div}; \mathcal{T}_h)$, respectively. Then, proceeding as in [3], we include the Galerkin-least squares terms given by

$$\frac{1}{2} \int\limits_{\Omega} (\nabla_h u_h - \boldsymbol{\sigma}_h) \cdot (\nabla_h v + \boldsymbol{\tau}) = 0 \quad \forall (\boldsymbol{\tau}, v) \in H(\operatorname{div}; \mathcal{T}_h) \times H^1(\mathcal{T}_h), \tag{7}$$

and

$$\int\limits_{\Omega} \operatorname{div}_h \boldsymbol{\sigma}_h \operatorname{div}_h \boldsymbol{\tau} = - \int\limits_{\Omega} f \operatorname{div}_h \boldsymbol{\tau} \quad \forall \boldsymbol{\tau} \in H(\operatorname{div}; \mathcal{T}_h), \tag{8}$$

and after adding (7), (8) and (5), we obtain the following discrete augmented discontinuous Galerkin formulation: *Find* $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h \times \mathcal{V}_h$ *such that*

$$A_{\mathrm{DG}}^{\mathrm{stab}}\big((\boldsymbol{\sigma}_h, u_h), (\boldsymbol{\tau}, v)\big) = F_{\mathrm{DG}}^{\mathrm{stab}}(\boldsymbol{\tau}, v) \quad \forall (\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma}_h \times \mathcal{V}_h, \tag{9}$$

where the (nonsymmetric) bilinear form

$$A_{\mathrm{DG}}^{\mathrm{stab}} : \big(H(\operatorname{div}; \mathcal{T}_h) \times H^1(\mathcal{T}_h)\big) \times \big(H(\operatorname{div}; \mathcal{T}_h) \times H^1(\mathcal{T}_h)\big) \to \mathbb{R}$$

and the linear functional

$$F_{\mathrm{DG}}^{\mathrm{stab}} : H(\operatorname{div}; \mathcal{T}_h) \times H^1(\mathcal{T}_h) \to \mathbb{R}$$

are defined by

$$A_{\mathrm{DG}}^{\mathrm{stab}}\big((\boldsymbol{\sigma}, w), (\boldsymbol{\tau}, v)\big) := \int\limits_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} + \int\limits_{\Omega} w \operatorname{div}_h \boldsymbol{\tau} - \int\limits_{\mathcal{E}_I \cup \mathcal{E}_N} [\![\boldsymbol{\tau}]\!] \big(\{w\} - \boldsymbol{\beta} \cdot [\![w]\!]\big) - \int\limits_{\Omega} v \operatorname{div}_h \boldsymbol{\sigma}$$
$$+ \int\limits_{\mathcal{E}_I \cup \mathcal{E}_N} [\![\boldsymbol{\sigma}]\!] \big(\{v\} - \boldsymbol{\beta} \cdot [\![v]\!]\big) + \boldsymbol{\alpha}(w, v) + \frac{1}{2} \int\limits_{\Omega} (\nabla_h w - \boldsymbol{\sigma}) \cdot (\nabla_h v + \boldsymbol{\tau}) + \int\limits_{\Omega} \operatorname{div}_h \boldsymbol{\sigma} \operatorname{div}_h \boldsymbol{\tau},$$

and

$$F_{\mathrm{DG}}^{\mathrm{stab}}(\boldsymbol{\tau}, v) := \int\limits_{\Omega} f v + \int\limits_{\mathcal{E}_D} \alpha g_D v + \int\limits_{\mathcal{E}_N} g_N v + \int\limits_{\mathcal{E}_D} g_D \boldsymbol{\tau} \cdot \boldsymbol{\nu} - \int\limits_{\Omega} f \operatorname{div}_h \boldsymbol{\tau},$$

for all $(\boldsymbol{\sigma}, w), (\boldsymbol{\tau}, v) \in H(\operatorname{div}; \mathcal{T}_h) \times H^1(\mathcal{T}_h)$. Therefore, we introduce $\boldsymbol{\Sigma}_h$ and $\mathcal{V}_h$ as

$$\boldsymbol{\Sigma}_h := \big\{\boldsymbol{\sigma}_h \in H(\operatorname{div}; \mathcal{T}_h) \colon \ \boldsymbol{\sigma}_h|_T \in RT_r(T) \ \forall T \in \mathcal{T}_h\big\},$$
$$\mathcal{V}_h := \big\{v_h \in L^2(\Omega) \colon \ v_h|_T \in \mathbf{P}_k(T) \ \forall T \in \mathcal{T}_h\big\}$$

with $k \geqslant 1$ and $r \geqslant 0$. Hereafter, given an integer $\kappa \geq 0$ we denote by $\mathbf{P}_\kappa(T)$ the space of polynomials of degree at most κ on T, and for each $T \in \mathcal{T}_h$ we introduce the local Raviart–Thomas space of order κ (cf. [10]), $RT_\kappa(T) := [\mathbf{P}_\kappa(T)]^2 \oplus [x]\mathbf{P}_\kappa(T) \subseteq [\mathbf{P}_{\kappa+1}(T)]^2$.

In addition, the space $\boldsymbol{\Sigma}_h$ is provided with the usual product norm of $\boldsymbol{\Sigma} := H(\mathrm{div}; \mathcal{T}_h)$, which is denoted by $\|\cdot\|_{\boldsymbol{\Sigma}}$, that is

$$\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}}^2 := \|\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\mathrm{div}_h\,\boldsymbol{\tau}\|_{0,\Omega}^2 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma},$$

while for $\boldsymbol{\mathcal{V}}_h$ we introduce its seminorm $|\cdot|_h : H^1(\mathcal{T}_h) \to \mathbb{R}$ and energy norm $|||\cdot|||_h : H^1(\mathcal{T}_h) \to \mathbb{R}$ as

$$|v|_h^2 := \big\|\alpha^{1/2}[\![v]\!]\big\|_{\mathcal{E}_I \cup \mathcal{E}_D}^2 \quad \forall v \in H^1(\mathcal{T}_h), \quad \text{and} \quad |||v|||_h^2 := \|\nabla_h v\|_{0,\Omega}^2 + |v|_h^2 \quad \forall v \in H^1(\mathcal{T}_h),$$

respectively. In addition, we define $\|(\cdot,\cdot)\|_{\mathrm{DG}} : H(\mathrm{div}; \mathcal{T}_h) \times H^1(\mathcal{T}_h) \to \mathbb{R}$ by

$$\big\|(\boldsymbol{\tau}, v)\big\|_{\mathrm{DG}}^2 := \|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}}^2 + |||v|||_h^2 \quad \forall (\boldsymbol{\tau}, v) \in H(\mathrm{div}; \mathcal{T}_h) \times H^1(\mathcal{T}_h).$$

Lemma 2.1. *There exist positive constants C_{A} and C_{F}, independent of the meshsize, such that*

$$\big|A_{\mathrm{DG}}^{\mathrm{stab}}\big((\boldsymbol{\sigma}, w), (\boldsymbol{\tau}, v)\big)\big| \leqslant C_{\mathrm{A}} \big\|(\boldsymbol{\sigma}, w)\big\|_{\mathrm{DG}} \big\|(\boldsymbol{\tau}, v)\big\|_{\mathrm{DG}}, \tag{10}$$

$$A_{\mathrm{DG}}^{\mathrm{stab}}\big((\boldsymbol{\tau}, v), (\boldsymbol{\tau}, v)\big) \geqslant \frac{1}{2} \big\|(\boldsymbol{\tau}, v)\big\|_{\mathrm{DG}}^2, \tag{11}$$

$$\big|F_{\mathrm{DG}}^{\mathrm{stab}}(\boldsymbol{\tau}, v)\big| \leqslant C_{\mathrm{F}} \mathcal{B}(f, g_D, g_N) \big\|(\boldsymbol{\tau}, v)\big\|_{\mathrm{DG}}, \tag{12}$$

for all $(\boldsymbol{\sigma}, w), (\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma}_h \times H^1(\mathcal{T}_h)$, where

$$\mathcal{B}(f, g_D, g_N) := \big(\|f\|_{0,\Omega}^2 + \|\alpha^{1/2} g_D\|_{0,\mathcal{E}_D}^2 + \|\alpha^{1/2} g_N\|_{0,\mathcal{E}_N}^2\big)^{1/2}.$$

Proof. The results are consequence of (6), Cauchy–Schwarz inequality and Lemma 3.1 in [5]. We omit further details. □

Theorem 2.1. *Problem* (9) *is uniquely solvable and such that there hold*

$$\big\|(\boldsymbol{\sigma}_h, u_h)\big\|_{\mathrm{DG}} \leqslant 2 C_{\mathrm{F}} \mathcal{B}(f, g_D, g_N), \tag{13}$$

and

$$\big\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)\big\|_{\mathrm{DG}} \leqslant 2 C_{\mathrm{A}} \inf_{(\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\mathcal{V}}_h} \big\|(\boldsymbol{\sigma} - \boldsymbol{\tau}, u - v)\big\|_{\mathrm{DG}}, \tag{14}$$

with $(\boldsymbol{\sigma}, u) \in H(\mathrm{div}; \Omega) \times H^1(\Omega)$ being the exact solution of (2).

Proof. The existence and uniqueness of the solution of (9), as well as (13), are guaranteed thanks to Lemma 2.1 by applying Lax–Milgram lemma. Finally, the derivation of the Céa estimate (14) follows straightforward since formulation (9) is consistent with the exact solution $(\boldsymbol{\sigma}, u)$, that is

$$A_{\mathrm{DG}}^{\mathrm{stab}}\big((\boldsymbol{\sigma}, u), (\boldsymbol{\tau}, v)\big) = F_{\mathrm{DG}}^{\mathrm{stab}}(\boldsymbol{\tau}, v) \quad \forall (\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\mathcal{V}}_h. \qquad \square$$

3. A priori error estimates

In order to obtain the a priori error estimates for the scheme (9) we need the following lemmas, which establish local approximation properties of piecewise polynomials approximations of $H^1(T)$ and $H(\mathrm{div}; T)$.

Now, for each $T \in \mathcal{T}_h$ and a positive integer k, we consider a local interpolation operator $\mathcal{E}_T^k : [H^1(T)]^2 \to RT_{k-1}(T)$ satisfying

$$\mathrm{div}(\boldsymbol{\tau}) = 0 \quad \Rightarrow \quad \mathrm{div}\big(\mathcal{E}_T^k(\boldsymbol{\tau})\big) = 0 \quad \forall \boldsymbol{\tau} \in \big[H^1(T)\big]^2. \tag{15}$$

In addition, thanks to Theorem 6.1 in [10], $\mathcal{E}_T^k$ verifies

$$\mathrm{div}\big(\mathcal{E}_T^k(\boldsymbol{\tau})\big) = \Pi_T^{k-1}\big(\mathrm{div}(\boldsymbol{\tau})\big) \quad \forall \boldsymbol{\tau} \in \big[H^1(T)\big]^2. \tag{16}$$

Then, we can establish the following approximation results.

Lemma 3.1. *For any $\boldsymbol{\tau} \in [H^l(T)]^2$ with $\operatorname{div}(\boldsymbol{\tau}) \in H^s(T)$, where l and s are nonnegative integers, respectively, there exists $C > 0$, independent of the meshsize, such that*

$$\left\|\boldsymbol{\tau} - \mathcal{E}_T^k(\boldsymbol{\tau})\right\|_{0,T} \leqslant C h_T^l |\boldsymbol{\tau}|_{l,T}, \quad 1 \leqslant l \leqslant r, \tag{17}$$

and

$$\left\|\operatorname{div}\left(\boldsymbol{\tau} - \mathcal{E}_T^k(\boldsymbol{\tau})\right)\right\|_{0,T} \leqslant C h_T^s \left|\operatorname{div}(\boldsymbol{\tau})\right|_{s,T}, \quad 0 \leqslant s \leqslant r. \tag{18}$$

Proof. It follows from the proof of Theorem 6.3 in [10], making use of Theorems 3.1 in [7]. □

The main result of this work is described next.

Theorem 3.1. *Let $(\boldsymbol{\sigma}, u)$ and $(\boldsymbol{\sigma}_h, u_h)$ be the unique solutions of (2) and (9), respectively. Then, assuming that $u|_T \in H^{t+1}(T)$, $\boldsymbol{\sigma}|_T \in [H^t(T)]^2$, and $\operatorname{div}(\boldsymbol{\sigma}|_T) \in H^s(T)$ with $t \geqslant 1$, and $0 \leqslant s \leqslant t$, for all $T \in \mathcal{T}_h$, there exists $C_{\mathtt{err}} > 0$, independent of the meshsize, such that*

$$\left\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)\right\|_{\mathrm{DG}}^2 \leqslant C_{\mathtt{err}} \sum_{T \in \mathcal{T}_h} h_T^{2\min\{t,s,k,r+1\}} \left\{\|\boldsymbol{\sigma}\|_{t,T}^2 + \|u\|_{t+1,T}^2 + \left\|\operatorname{div}(\boldsymbol{\sigma})\right\|_{s,T}^2\right\}. \tag{19}$$

Proof. It is consequence of Céa estimate (cf. Theorem 2.1), Theorem 3.1 in [7], and Lemma 3.1. □

Finally, we remark that the corresponding a-posteriori error analysis, further details on the remaining proofs, as well as some numerical examples validating our results, will be reported in [2].

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