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C. R. Acad. Sci. Paris, Ser. I 344 (2007) 299–304
COMPTES RENDUS MATHEMATIQUE
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Partial Differential Equations

Maximal solutions of the equation $\Delta u = u^q$ in arbitrary domains

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Received and accepted 1 January 2007

Available online 9 February 2007

Presented by Haïm Brezis

Abstract

We prove bilateral capacitary estimates for the maximal solution U_F of $-\Delta u + u^q = 0$ in the complement of an arbitrary closed set $F \subset \mathbb{R}^N$, involving the Bessel capacity $C_{2,q'}$, for q in the supercritical range $q \geqslant q_c := N/(N-2)$. We derive a pointwise necessary and sufficient condition, via a Wiener type criterion, in order that $U_F(x) \to \infty$ as $x \to y$ for given $y \in \partial F$. Finally we prove a general uniqueness result for large solutions. ***To cite this article: M. Marcus, L. Véron, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Solutions maximales de $\Delta u = u^q$ dans un domaine arbitraire. Nous démontrons une estimation capacitaire bilatérale de la solution maximale U_F de $-\Delta u + u^q = 0$ dans un domaine quelconque de $\mathbb{R}^N$ impliquant la capacité de Bessel $C_{2,q'}$ dans le cas sur-critique $q \geqslant q_c := N/(N-2)$. Grâce à un critère de type Wiener, nous en déduisons une condition nécessaire et suffisante pour que cette solution maximale tende vers l'infini en un point du bord du domaine. Finalement nous prouvons un résultat général d'unicité des grandes solutions. ***Pour citer cet article : M. Marcus, L. Véron, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Soit F un sous-ensemble compact non-vide de $\mathbb{R}^N$ de complémentraire F^c connexe et $q > 1$. Il est bien connu qu'il existe une solution maximale U_F de

$$-\Delta u + u^q = 0, \tag{1}$$

dans $F^c = \mathbb{R}^N \setminus F$. En outre $U_F = 0$ si et seulement si $C_{2,q'}(F) = 0$, où $q' = q/(q-1)$ et $C_{2,q'}$ désigne la capacité de Bessel en dimension N [2]. Si $1 < q < q_c := N/(N-2)$, la capacité de tout point est positive et la solution maximale est une grande solution [9], c'est-à-dire vérifie

$$\lim_{F^c \ni x \to y} U_F(x) = \infty, \tag{2}$$

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2007.01.002

pour tout $y \in \partial F^c$, et la relation (2) est uniforme en y. En outre U_F est l'unique grande solution si on suppose $\partial F^c \subset \partial \overline{F^c}$. Dans le cas sur-critique $q \geqslant q_c$ la situation est beaucoup plus compliquée dans la mesure où les singularités isolées sont éliminables et où il existe une grande variété de solutions. Si $q = 2$, $N \geqslant 3$, Dhersin et Le Gall [3] ont obtenu, par des méthodes probabilistes, des estimations précises portant sur U_F et utilisant la capacité $C_{2,2}$. De leurs estimations découle une condition nécessaire et suffisante, exprimée par un critère du type de Wiener, pour que U_F vérifie (2) en un point $y \in \partial F^c$.

Labutin [4] a réussi à étendre partiellement les résultats de [3] dans le cas $q \geqslant q_c$. Plus précisément il a prouvé que U_F est une grande solution si et seulement si le critère de Wiener de [3], avec $C_{2,2}$ remplacé par $C_{2,q'}$, est vérifié en *tout* point de ∂F^c, cependant il n'obtient pas l'estimation ponctuelle (2). Les estimations de Labutin sont optimales si $q > q_c$, mais pas si $q = q_c$. Dans cette Note nous étendons les résultats de [3] par des méthodes purement analytiques.

Si F est un sous-ensemble fermé non vide de $\mathbb{R}^N$, $x \in \mathbb{R}^N$ et $m \in \mathbb{Z}$ nous notons

$$T_m(x) = \left\{y \in \mathbb{R}^N \colon\ 2^{-m-1} \leqslant |x-y| \leqslant 2^{-m}\right\}$$
$$F_m(x) = F \cap T_m(x) \quad \text{et} \quad F_m^*(x) = F \cap \overline{B}_{2^{-m}}(x).$$

On définit le *potentiel $C_{2,q'}$-capacitaire W_F* de F par

$$W_F(x) = \sum_{-\infty}^{\infty} 2^{2m/(q-1)} C_{2,q'}\big(2^m F_m(x)\big). \tag{3}$$

Théorème 1. *Il existe une constante $c = c(N,q) > 0$ telle que*

$$cW_F(x) \leqslant U_F(x) \leqslant \frac{1}{c} W_F(x) \quad \forall x \in F^c. \tag{4}$$

Pour $q > q_c$ cette estimation est la même que celle de Labutin. Notre démonstration est inspirée de la sienne tout en faisant intervenir des arguments nouveaux qui simplifient notoirement sa démarche. En utilisant la définition de la capacité de Bessel on démontre alors que la fonction W_F est semi-continue inférieurement dans $\overline{F^c}$. On en déduit

Théorème 2. *Pour tout point $y \in \partial F^c$,*

$$\lim_{F^c \ni x \to y} U_F(x) = \infty \iff W_F(y) = \infty. \tag{5}$$

Par suite U_F est une grande solution si et seulement si $W_F(y) = \infty$ pour tout $y \in \partial F^c$.

Il est facile de vérifier que si $W_F(y) = \infty$, alors y est un point épais de F, au sens de la topologie fine $\mathfrak{T}_q$ associée à la capacité $C_{2,q'}$. En utilisant la propriété de Kellog [1] que vérifie la capacité $C_{2,q'}$, on en déduit que la solution maximale U_F est une *presque grande solution* dans le sens suivant : La relation (2) a lieu sauf peut-être sur un ensemble de ∂F^c de capacité $C_{2,q'}$ nulle.

Il est classique que l'équation $-\Delta u + |u|^{q-1}u = \mu$ admet une unique solution, notée u_μ, pour tout $\mu \in W^{-2,q'}(\mathbb{R}^N)$ [2]. On a alors le résultat suivant :

Théorème 3. *Pour tout sous-ensemble fermé $F \subset \mathbb{R}^N$,*

$$U_F = \sup\big\{u_\mu \colon\ \mu \in W^{-2,q'}\big(\mathbb{R}^N\big), \mu\big(F^c\big) = 0\big\}. \tag{6}$$

Par suite U_F est σ-modérée, c'est-à-dire qu'il existe une suite croissante $\{\mu_n\} \subset W^{-2,q'}(\mathbb{R}^N)$ telle que $\mu_n(F^c) = 0$ et $u_{\mu_n} \uparrow u$.

Cet énoncé est l'analogue dans le cas du problème elliptique intérieur de résultats similaires concernant les problèmes elliptique au bord [7] et parabolique [8]. Enfin, nous avons le résultat d'unicité suivant où nous désignons par $\widetilde{E}$ la fermeture de $E \subset \mathbb{R}^N$ pour la topologie $\mathfrak{T}_q$.

Théorème 4. *Pour tout ouvert non vide $D \subset \mathbb{R}^N$, posons $F = D^c$ et $F_0 = \widetilde{D}^c$ (c'est-à-dire que F_0 est l'intérieur de F pour la topologie $\mathfrak{T}_q$). Si $C_{2,q'}(F \setminus \widetilde{F}_0) = 0$, alors il existe au plus une grande solution de* (1) *dans D.*

1. Introduction

In this Note we study positive solutions of the equation

$$-\Delta u + u^q = 0, \tag{1}$$

in $\mathbb{R}^N \setminus F$, $N \geqslant 3$, where F is a non-empty compact set with F^c connected and $q > 1$. More precisely, we shall study the behavior of the maximal solution of this problem, which we denote by U_F. The existence of the maximal solution is guaranteed by the Keller–Osserman estimates (see [6] for discussion about large solutions and the references therein). It is known [2] that, if $C_{2,q'}(F) = 0$ then $U_F = 0$. If u is a solution of (1) in $D = \mathbb{R}^N \setminus F$ and u blows up at every point of ∂D we say that u is a *large solution* in D. Obviously a large solution exists in D if and only if U_F is a large solution.

Our aim is: (a) to provide a necessary and sufficient condition for the blow up of U_F at an arbitrary point $y \in F$; and (b) to obtain a general uniqueness result for large solutions.

In the subcritical case, i.e. $1 < q < q_c := N/(N-2)$, these problems are well understood. In this case $C_{2,q'}(F) > 0$ for any non-empty set and it is classical that positive solutions may have isolated point singularities of two types: weak and strong. This easily implies that the maximal solution U_F is always a large solution in D. In addition it is proved in [9] that the large solution is unique if it is assumed $\partial F^c \subset \partial \overline{F^c}^c$.

In the supercritical case, i.e. $q \geqslant q_c$, the situation is much more complicated. In this case point singularities are removable and there exists a large variety of singular solutions.

Sharp estimates for U_F were obtained by Dhersin and Le Gall [3] in the case $q = 2$, $N \geqslant 3$. These estimates were expressed in terms of the Bessel capacity $C_{2,2}$ and were used to provide a Wiener type criterion for the pointwise blow up of U_F, i.e., for $y \in F$,

$$\lim_{F^c \ni x \to y} U_F(x) = \infty \iff \text{the Wiener type criterion is satisfied at } y. \tag{2}$$

These results were obtained by probabilistic tools; hence the restriction to $q = 2$.

Labutin [4] succeeded in partially extending the results of [3] to $q \geqslant q_c$. Specifically, he proved that U_F is a large solution if and only if the Wiener criterion of [3], with $C_{2,2}$ replaced by $C_{2,q'}$, is satisfied *at every point of* F. The pointwise blow up was not established. Labutin's result was obtained by analytic techniques. As in [3], the proof is based on upper and lower estimates for U_F, in terms of the capacity $C_{2,q'}$. Labutin's estimates are sharp for $q > q_c$ but not for $q = q_c$.

Conditions for uniqueness of large solutions, for arbitrary $q > 1$, can be found in [6] and references therein.

In the present Note we obtain a full extension of the results of [3] to $q \geqslant q_c$, $N \geqslant 3$.

Further we establish the following rather surprising fact: For any non-empty closed set $F \subsetneq \mathbb{R}^N$, the maximal solution U_F is an 'almost large' solution in D in the following sense: (2) holds at all points of F with the possible exception of a set of $C_{2,q'}$-capacity zero. (Of course if y is an interior point of F, (2) holds in void.)

Finally we provide a capacitary sufficient condition for the uniqueness of large solutions.

2. Statement of main results

Throughout the remainder of the Note we assume that $q \geqslant q_c$. We start with some notation. For any set $A \subset \mathbb{R}^N$ we denote by ρ_A the distance function, $\rho_A(x) = \operatorname{dist}(x, A)$ for every $x \in \mathbb{R}^N$. If F is a closed set and $x \in \mathbb{R}^N$ we denote

$$\begin{aligned} &T_m(x) = \left\{ y \in \mathbb{R}^N \colon 2^{-(m+1)} \leqslant |y - x| \leqslant 2^{-m} \right\}, \\ &F_m(x) = F \cap T_m(x), \qquad F_m^*(x) = F \cap \overline{B}_{2^{-m}}(x). \end{aligned} \tag{3}$$

As usual $C_{\alpha,p}$ denotes Bessel capacity in $\mathbb{R}^N$. Note that if $\alpha = 2$ and $p = q' = q/(q-1)$ then, for $q \geqslant N/(N-2)$, $\alpha p \leqslant N$. Put

$$W_F(x) = \sum_{-\infty}^{\infty} 2^{2m/(q-1)} C_{2,q'}\big(2^m F_m(x)\big). \tag{4}$$

W_F is called the $C_{2,q'}$-*capacitary potential* of F.

Observe that $2^m F_m^*(x) \subset B_1(x)$ and that, for every $x \in F^c$, there exists a minimal integer $M(x)$ such that $F_m(x) = \emptyset$ for $M(x) < m$. Therefore

$$W_F(x) = \sum_{-\infty}^{M(x)} 2^{2m/(q-1)} C_{2,q'}\bigl(2^m F_m(x)\bigr) < \infty \quad \forall x \in F^c. \tag{5}$$

It is known that there exists a constant C depending only on q, N such that

$$W_F(x) \leqslant W_F^*(x) := \sum_{m(x)}^{\infty} 2^{-2m/(q-1)} C_{2,q'}\bigl(2^{-m} F_m^*(x)\bigr) \leqslant C W_F(x) \tag{6}$$

for every $x \in F^c$, see e.g. [7].

In the following results F denotes a proper closed subset of $\mathbb{R}^N$. The first theorem describes the capacitary estimates for the maximal solution.

Theorem 2.1. *The maximal solution U_F satisfies the inequalities*

$$\frac{1}{c} W_F(x) \leqslant U_F(x) \leqslant c W_F(x) \quad \forall x \in F^c. \tag{7}$$

For $q > q_c$ these estimates are equivalent to those obtained by Labutin [4]. Our proof is inspired by the proof of [4], but employs some new arguments which lead to a sharp estimate in the border case $q = q_c$ as well. Using the previous theorem we establish:

Theorem 2.2. *For every point $y \in F$,*

$$\lim_{F^c \ni x \to y} U_F(x) = \infty \iff W_F(y) = \infty. \tag{8}$$

Consequently U_F is a large solution in F^c if and only if $W_F(y) = \infty$ for every $y \in F$.

Theorem 2.3. *For any closed set $F \subsetneq \mathbb{R}^N$, the maximal solution U_F is an almost large solution in $D = F^c$ (see the definition of this term in the introduction).*

It is known [2] that if $\mu \in W_+^{-2,q}(\mathbb{R}^N)$ there exists a unique solution of the equation $-\Delta u + u^q = \mu$ in $\mathbb{R}^N$. This solution will be denoted by u_μ.

Theorem 2.4. *For any closed set $F \subsetneq \mathbb{R}^N$,*

$$U_F = \sup\bigl\{u_\mu:\ \mu \in W_+^{-2,q}\bigl(\mathbb{R}^N\bigr),\ \mu\bigl(F^c\bigr) = 0\bigr\}. \tag{9}$$

Thus U_F is σ-moderate, i.e., there exists an increasing sequence $\{\mu_n\} \subset W_+^{-2,q}(\mathbb{R}^N)$ such that $\mu_n(F^c) = 0$ and $u_{\mu_n} \uparrow U_F$.

For the next result we need the concept of the $C_{2,q'}$-fine topology (in $\mathbb{R}^N$) that we shall denote by $\mathfrak{T}_q$. For its definition and basic properties see [1, Ch. 6]. The closure of a set E in the topology $\mathfrak{T}_q$ will be denoted by $\widetilde{E}$. The following uniqueness result holds:

Theorem 2.5. *Let $D \subset \mathbb{R}^N$ be a non-empty, bounded open set. Put $F = D^c$ and $F_0 = (\widetilde{D})^c$ so that F_0 is the $\mathfrak{T}_q$-interior of F. If $C_{2,q'}(F \setminus \widetilde{F}_0) = 0$ then there exists at most one large solution in D.*

3. Sketch of proofs

On the proof of Theorem 2.1. The proof of this theorem is an adaptation of the proof of the capacitary estimates for boundary value problems in [7]. A central element of the proof in that paper is the mapping $\mathbb{P}: W_+^{-2/q,q}(\partial\Omega) \mapsto$

$L^q(\Omega;\rho_{\partial\Omega})$ given by $\mathbb{P}(\mu)=\int_{\partial\Omega}P(x,y)\,\mathrm{d}\mu(y)$ where P is the Poisson kernel in Ω. In the proof of the present result the same role is played by the Green operator acting on bounded measures in $\mathbb{R}^N$. □

On the proof of Theorem 2.2. Denote

$$a_m(x)=C_{2,q'}\big(2^m F_m(x)\big),\quad a_m^*(x)=C_{2,q'}\big(2^m F_m^*(x)\big). \tag{10}$$

First we show that $W_F(y)=\infty$ implies that $\lim_{D\ni x\to y}U_F(x)=\infty$. Let $x\in F^c$ and let λ be an integer such that $2^{-\lambda}\leqslant|x-y|\leqslant 2^{-\lambda+1}$. Obviously $\lambda\leqslant M(x)$. For $m\leqslant\lambda$:

$$a_m^*(y)\leqslant C_{2,q'}\big(2\big(2^{m-1}F^*{}_m(x)\big)\big)\leqslant ca_{m-1}^*(x).$$

Therefore

$$\sum_1^\lambda 2^{2m/(q-1)}a_m^*(y)\leqslant c\sum_1^\lambda 2^{2m/(q-1)}a_{m-1}^*(x)\leqslant c\sum_0^\lambda 2^{2m/(q-1)}a_m^*(x)\leqslant W_F^*(x).$$

As $x\to y$, $\lambda\to\infty$ and the left-hand side tends to ∞. The reverse implication is a consequence of the following property of W_F:

Lemma 3.1. *The function $y\mapsto W_F^*(y)$ is lower semicontinuous on $\overline{F^c}$ In addition, if $W_F^*(y)<\infty$ then*

$$\liminf_{x\to y}W_F^*(x)<\infty.$$

For proving this result, we use the fact that, for any $y\in\overline{F^c}$, and any $m\in\mathbb{Z}$,

$$C_{2,q'}\big(2^mF_m^*(y)\big)=\inf\big\{\|\zeta\|^{q'}_{W^{2,q'}}:\ \zeta\in C_0^\infty\big(\mathbb{R}^N\big):\ \zeta\geqslant 0,\zeta\geqslant 1\text{ in a neighborhood of }2^mF_m^*(y)\big\}.$$

Thus, if $\zeta\geqslant 1$ in a neighborhood of $2^mF_m^*(y)$, it implies that, for $|x-y|$ small enough, $\zeta\geqslant 1$ in a neighborhood of $2^mF_m^*(x)$. This implies

$$\lim_{\epsilon\to 0}\sup_{x\in\overline{F^c}\cap B_\epsilon(y)}C_{2,q'}\big(2^mF_m^*(x)\big)=\limsup_{\overline{F^c}\ni x\to y}C_{2,q'}\big(2^mF_m^*(x)\big)\leqslant C_{2,q'}\big(2^mF_m^*(y)\big).$$

By a similar argument, one obtains $C_{2,q'}(2^mF_m^*(y))\leqslant\liminf_{x\to y}C_{2,q'}(2^mF_{m-1}^*(x))$. This implies the first assertion. The second is proved by an argument involving the quasi additivity of capacity. □

On the proof of Theorem 2.3. It is not difficult to verify that, if x is a thick point of F in the topology $\mathfrak{T}_q$ (or $\mathfrak{T}_q$-thick point), then $W_F(x)=\infty$. (For the definition of a thick point in a fine topology and the properties stated below see [1, Ch. 6].) The set of $\mathfrak{T}_q$-thick points of F is denoted by $b_q(F)$ and it is known that, if F is $\mathfrak{T}_q$-closed then $b_q(F)\subset F$ and $C_{2,q'}(F\setminus b_q(F))=0$ (this is called the Kellog property). Of course any set closed in the Euclidean topology is $\mathfrak{T}_q$-closed. Therefore, by Theorem 2.1, U_F blows up $C_{2,q'}$-a.e. on ∂D. □

On the proof of Theorem 2.4. Let us denote the right-hand side of (9) by V_F. Obviously $V_F\leqslant U_F$ and the proof of Theorem 2.1 actually shows that $\frac{1}{c}W_F(x)\leqslant V_F(x)$. Therefore $U_F\leqslant CV_F$ where C is a constant depending only on N,q. By an argument introduced in [5] this implies that $U_F=V_F$. □

On the proof of Theorem 2.5. The proof is based on the following:

Lemma 3.2. *If $C_{2,q'}(F\setminus\widetilde{F}_0)=0$ then*

$$U_F=\sup\big\{u_\mu:\ \mu\in W_+^{-2,q}(\mathbb{R}^N),\operatorname{supp}\mu\subset F_0\big\}.$$

The proof of the lemma involves subtle properties of the $C_{2,q'}$-fine topology.

The lemma implies that for every $x\in D$ there exists $\mu\in W_+^{-2,q}(\mathbb{R}^N)$ such that $\operatorname{supp}\mu$ is a compact subset of F_0 and $U_F(x)\leqslant Cu_\mu(x)$. Suppose that u is a large solution in D. Since u_μ is bounded in ∂D it follows that $u_\mu<u$. Thus $U_F\leqslant Cu$. By the argument of [5] mentioned before, this implies that $u=U_F$.

Acknowledgements

Both authors were partially sponsored by an EC grant through the RTN Program 'Front-Singularities', HPRN-CT-2002-00274 and by the French–Israeli cooperation program through grant No. 3-1352. The first author (M.M.) also wishes to acknowledge the support of the Israeli Science Foundation through grant No. 145-05.

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