Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 344 (2007) 487–492
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Partial Differential Equations

Uniqueness results for pseudomonotone problems with $p > 2$

Juan Casado-Díaz [a], François Murat [b], Alessio Porretta [c]

[a] Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, c/Tarfia s/n, 41012 Sevilla, Spain
[b] Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, boîte courrier 187, 75252 Paris cedex 05, France
[c] Dipartimento di Matematica, Università di Roma Tor Vergata, Via della ricerca scientifica 1, 00133 Roma, Italy

Received 5 April 2005; accepted 30 January 2007

Presented by Pierre-Louis Lions

Abstract

We consider a pseudomonotone operator, the model of which is $-\operatorname{div}(b(x,u)|\nabla u|^{p-2}\nabla u)$ with $1 < p < +\infty$ and $b(x,s)$ a Lipschitz continuous function in s which hold satisfies $0 < \alpha \leqslant b(x,s) \leqslant \beta < +\infty$. We show that the comparison principle (and therefore the uniqueness for the Dirichlet problem) in two particular cases, namely the one-dimensional case, and the case where at least one of the right-hand sides does not change sign. To the best of our knowledge these results are new for $p > 2$. Full detailed proofs are given in the present Note. The results continue to hold when Ω is unbounded. ***To cite this article: J. Casado-Díaz et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Résultats d'unicité pour des problèmes pseudomonotones avec $p > 2$. Nous considérons un opérateur pseudomonotone du type $-\operatorname{div}(b(x,u)|\nabla u|^{p-2}\nabla u)$, avec $1 < p < +\infty$ et $b(x,s)$ une fonction Lipschitzienne en s qui vérifie $0 < \alpha \leqslant b(x,s) \leqslant \beta < +\infty$. Nous démontrons que cet opérateur satisfait le principe de comparaison (et donc qu'on a unicité pour le problème de Dirichlet) dans deux cas particuliers : en dimension 1, et dans le cas où au moins l'un des deux seconds membres ne change pas de signe. A notre connaissance, ces résultats sont nouveaux quand $p > 2$. Les démonstrations complètes sont données dans cette Note. Les résultats restent valides quand Ω est non borné. ***Pour citer cet article : J. Casado-Díaz et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Dans cette Note, nous démontrons deux résultats d'unicité pour la solution du problème de Dirichlet pseudomonotone

$$u - U_0 \in W_0^{1,p}(\Omega), \quad \int_\Omega a(x,u,\nabla u)\nabla v \,\mathrm{d}x = \langle f, v\rangle, \quad \forall v \in W_0^{1,p}(\Omega), \tag{1}$$

E-mail addresses: jcasadod@us.es (J. Casado-Díaz), murat@ann.jussieu.fr (F. Murat), porretta@mat.uniroma2.it (A. Porretta).

1631-073X/$ – see front matter © 2007 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.crma.2007.02.007

quand Ω est un ouvert borné de $\mathbb{R}^N$ (aucune hypothèse de régularité n'est faite sur le bord de $\partial\Omega$), quand $1 < p < +\infty$, et quand la fonction $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ est une fonction de Carathéodory qui vérifie les hypothèses (6), (7) et (8) de la version anglaise.

De façon précise, nous démontrons sous ces hypothèses le principe de comparaison, c'est-à-dire que les hypothèses (3) et (4) de la version anglaise impliquent (5) (ce qui évidemment entraîne l'unicité de la solution de (1)) dans les deux cas suivants : le cas où $N = 1$, et le cas où dans (3), f_1 ou f_2 est de signe constant. Un résultat plus fort est connu quand $1 < p \leqslant 2$ (voir [2][1]), à savoir le principe de comparaison (et donc l'unicité) dans le cas où la fonction a vérifie les hypothèses (6) et (2) de la version anglaise et une hypothèse de coercivité, et ce sans restriction sur la dimension ni sur le signe du second membre. En revanche nos résultats semblent nouveaux quand $p > 2$. Les démonstrations, qui sont intégralement données dans la Section 2 de cette Note, reposent sur deux lemmes (Lemma 2.1 et Lemma 2.2 de la version anglaise) qui ont leur intérêt propre. Ils utilisent de façon essentielle l'hypothèse (8), qui n'est pas l'hypothèse de croissance classiquement faite sur la fonction a. Enfin dans la brève Section 3, nous expliquons comment modifier nos énoncés pour que le principe de comparaison et les résultats d'unicité restent valables, avec les mêmes démonstrations, dans le cas d'ouverts non bornés.

1. Introduction and main results

In this Note we prove two uniqueness results for the solution of the Dirichlet pseudomonotone problem

$$u - U_0 \in W_0^{1,p}(\Omega), \quad \int_\Omega a(x, u, \nabla u)\nabla v \, dx = \langle f, v\rangle, \quad \forall v \in W_0^{1,p}(\Omega), \tag{1}$$

where Ω is a bounded open set of $\mathbb{R}^N$ (no smoothness assumption is made on $\partial\Omega$) (see Section 3 for the case where Ω is unbounded), $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function which satisfies coerciveness and growth conditions, so that the operator $u \to -\operatorname{div} a(x, u, \nabla u)$ is pseudomonotone in the Sobolev space $W_0^{1,p}(\Omega)$, f belongs to $W^{-1,p'}(\Omega)$ and U_0 to $W^{1,p}(\Omega)$. The existence of at least one solution for this problem is known since the celebrated result of J. Leray and J.-L. Lions (see [10,11]). When $1 < p \leqslant 2$, and when the function $a(x, s, \xi)$ is strongly monotone in ξ (condition (6) below) and satisfies the (weighted) Lipschitz continuity condition in s

$$\left|a(x, s, \xi) - a(x, s', \xi)\right| \leqslant \left(\gamma|\xi|^{p-1} + \gamma_0\left(|s| + |s'|\right)^{p-1} + \left|\ell(x)\right|\right)|s - s'|, \tag{2}$$

a.e. $x \in \Omega$, for every $s, s' \in \mathbb{R}$, for every $\xi \in \mathbb{R}^N$, with $\gamma > 0$, $\gamma_0 \geqslant 0$ and $\ell \in L^{p'}(\Omega)$, it is proved in [2][1], [9] (see also [1,4,5,12,13]) that the operator $u \to -\operatorname{div} a(x, u, \nabla u)$ satisfies the comparison principle in the following sense: when

$$u_i \in W^{1,p}(\Omega), \quad f_i \in W^{-1,p'}(\Omega), \quad -\operatorname{div} a(x, u_i, \nabla u_i) = f_i \quad \text{in } \mathcal{D}'(\Omega),\ i \in \{1, 2\}, \tag{3}$$

and when

$$f_1 \leqslant f_2 \quad \text{in } \Omega, \qquad u_1 \leqslant u_2 \ \text{ on } \partial\Omega, \tag{4}$$

(where the first assertion has to be understood in the sense of distributions or equivalently in the sense of $W^{-1,p'}(\Omega)$, and the second one as $(u_1 - u_2)^+ \in W_0^{1,p}(\Omega)$), then one has

$$u_1 \leqslant u_2 \quad \text{in } \Omega. \tag{5}$$

In particular, uniqueness holds for problem (1).

In the present paper, we assume $1 < p < +\infty$ (although it follows from the comparison principle stated just above that our results are not new when $1 < p \leqslant 2$) and that a satisfies the following assumptions: there exist $\alpha > 0$, $\beta > 0$, $\gamma > 0$ such that a.e. $x \in \Omega$, for every $s, s' \in \mathbb{R}$, for every $\xi, \xi' \in \mathbb{R}^N$,

[1] Let us explicitly note that there is a mistake in the writing of some of the assumptions in [2] : indeed there is no function which satisfies assumption (3) of [2] when $p < 2$ (just take $\xi \neq 0$ fixed and let η tend to zero). Nevertheless the results of [2] are correct and can be proved by (essentially) the proof used there, once assumptions (1) and (3) of [2] are replaced by $a(x, s, \xi)\xi \geqslant \alpha|\xi|^p - \gamma|s|^\sigma - \theta(x)$ with $\sigma < p$ and by assumption (6) of the present paper.

$$\big(a(x,s,\xi)-a(x,s,\xi')\big)(\xi-\xi')\geqslant\alpha\big(|\xi|+|\xi'|\big)^{p-2}|\xi-\xi'|^2, \tag{6}$$

$$\big|a(x,s,\xi)-a(x,s',\xi)\big|\leqslant\gamma|\xi|^{p-1}|s-s'|, \tag{7}$$

$$\big|a(x,s,\xi)\big|\leqslant\beta|\xi|^{p-1}. \tag{8}$$

Observe that assumption (7) is less general than (2), and that assumption (8) is less general than the growth condition which is classically assumed on a, namely $|a(x,s,\xi)|\leqslant\beta|\xi|^{p-1}+\beta_0|s|^{p-1}+|\ell(x)|$.

The model example for the function a is $a(x,s,\xi)=b(x,s)|\xi|^{p-2}\xi$, where b is a Carathéodory function which is Lipschitz continuous in s and satisfies $0<\alpha\leqslant b(x,s)\leqslant\beta<+\infty$. In the very special case where $b(x,s)=c(x)g(s)$, the comparison principle for (1) is easily proved by using the change of unknown function $v(x)=\int_0^{u(x)}g(t)^{\frac{1}{p-1}}\,\mathrm{d}t$, which transforms in this case the pseudomonotone problem (1) into a strongly monotone one. However, in our knowledge, if b does not have this special form and only satisfies assumptions (6), (7) and (8), the comparison principle (and uniqueness) for (1) is an open problem for $p>2$. Here we prove that the comparison principle and uniqueness hold in two particular cases: in the one-dimensional case, and when the sign of f_1 and/or f_2 is constant. Unfortunately the general case remains an open problem.

Theorem 1.1. *Assume $N=1$ and that* (6), (7) *and* (8) *hold. Let u_1,u_2,f_1,f_2 satisfy* (3) *and* (4). *Then $u_1\leqslant u_2$ a.e. in Ω. In particular, uniqueness holds for problem* (1) *when $N=1$.*

Theorem 1.2. *Assume that* (6), (7) *and* (8) *hold. Let u_1,u_2,f_1,f_2 satisfy* (3) *and* (4). *If the sign of f_1 and/or the sign of f_2 is constant in Ω, then $u_1\leqslant u_2$ a.e. in Ω. In particular, uniqueness holds for problem* (1) *when the sign of f is constant in Ω.*

2. Proofs

In this section we give complete proofs of the above results. We begin with two lemmas which have their own interest:

Lemma 2.1. *Assume that* (6), (7) *and* (8) *hold. Let u_1,u_2,f_1,f_2 satisfy* (3) *and* (4). *Then for every $\varepsilon>0$ we have*

$$\alpha\int\limits_{\{0<u_1-u_2<\varepsilon\}}\big(|\nabla u_1|+|\nabla u_2|\big)^{p-2}\big|\nabla(u_1-u_2)\big|^2\,\mathrm{d}x\leqslant\gamma\varepsilon\int\limits_{\{0<u_1-u_2<\varepsilon\}}\min\big\{|\nabla u_1|,|\nabla u_2|\big\}^{p-1}\big|\nabla(u_1-u_2)\big|\,\mathrm{d}x. \tag{9}$$

Moreover we have

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon^2}\int\limits_{\{0<u_1-u_2<\varepsilon\}}\big(|\nabla u_1|+|\nabla u_2|\big)^{p-2}\big|\nabla(u_1-u_2)\big|^2\,\mathrm{d}x=0. \tag{10}$$

Proof. For $\varepsilon>0$, we define $T_\varepsilon:\mathbb{R}\to\mathbb{R}$ as the usual truncation at height ε, namely

$$T_\varepsilon(s)=s\quad\text{if }|s|\leqslant\varepsilon,\qquad T_\varepsilon(s)=\varepsilon\,\mathrm{sgn}(s)\quad\text{if }|s|\geqslant\varepsilon. \tag{11}$$

Taking $T_\varepsilon(u_1-u_2)^+$ as test function in (3) for $i=1$ and $i=2$, making the difference, using $f_1\leqslant f_2$, denoting

$$U_1^\varepsilon=\big\{0<u_1-u_2<\varepsilon,\ \big|\nabla u_1(x)\big|\leqslant\big|\nabla u_2(x)\big|\big\},\qquad U_2^\varepsilon=\big\{0<u_1-u_2<\varepsilon,\ \big|\nabla u_2(x)\big|<\big|\nabla u_1(x)\big|\big\},$$

then adding and subtracting in the integral on U_1^ε the term $a(x,u_2,\nabla u_1)\nabla(u_1-u_2)$ and in the integral on U_2^ε the term $a(x,u_1,\nabla u_2)\nabla(u_1-u_2)$, we get, for every $\varepsilon>0$

$$\int\limits_{U_1^\varepsilon}\big(a(x,u_2,\nabla u_1)-a(x,u_2,\nabla u_2)\big)\nabla(u_1-u_2)\,\mathrm{d}x+\int\limits_{U_2^\varepsilon}\big(a(x,u_1,\nabla u_1)-a(x,u_1,\nabla u_2)\big)\nabla(u_1-u_2)\,\mathrm{d}x$$

$$\leqslant-\int\limits_{U_1^\varepsilon}\big(a(x,u_1,\nabla u_1)-a(x,u_2,\nabla u_1)\big)\nabla(u_1-u_2)\,\mathrm{d}x-\int\limits_{U_2^\varepsilon}\big(a(x,u_1,\nabla u_2)-a(x,u_2,\nabla u_2)\big)\nabla(u_1-u_2)\,\mathrm{d}x.$$

From (6) and (7) we obtain (9). Using in (9) $\min\{|\nabla u_1|, |\nabla u_2|\} \leqslant |\nabla u_1| + |\nabla u_2|$, Cauchy–Schwartz's inequality and the fact that the measure of the set $\{0 < u_1 - u_2 < \varepsilon\}$ tends to zero with ε yields (10). □

Lemma 2.2. *Assume that* (6), (7) *and* (8) *hold. Let* u_1, u_2, f_1, f_2 *satisfy* (3) *and* (4). *Then for* T_ε *defined by* (11) *and for every* $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, *we have*

$$\int_{\{u_2<u_1\}} a(x, u_i, \nabla u_i)\nabla\varphi \,\mathrm{d}x = \lim_{\varepsilon\to 0}\left\langle f_i, \frac{T_\varepsilon(u_1-u_2)^+}{\varepsilon}\varphi\right\rangle, \quad i \in \{1, 2\}. \tag{12}$$

Proof. For $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $i \in \{1, 2\}$, we take $\varphi\, T_\varepsilon(u_1 - u_2)^+/\varepsilon$ as test function in (3). This gives

$$\int_\Omega a(x, u_i, \nabla u_i)\nabla\varphi \frac{T_\varepsilon(u_1-u_2)^+}{\varepsilon}\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\{0<u_1-u_2<\varepsilon\}} a(x, u_i, \nabla u_i)\nabla(u_1-u_2)\,\varphi\,\mathrm{d}x = \left\langle f_i, \frac{T_\varepsilon(u_1-u_2)^+}{\varepsilon}\varphi\right\rangle. \tag{13}$$

We easily pass to the limit in the first term of (13) by using Lebesgue's dominated convergence theorem. For the second term of (13), using (8), $|\nabla u_i|^{p-1} \leqslant (|\nabla u_1| + |\nabla u_2|)^{p-1}$ and Cauchy–Schwartz's inequality, we get

$$\begin{aligned}\left|\frac{1}{\varepsilon}\int_{\{0<u_1-u_2<\varepsilon\}} a(x, u_i, \nabla u_i)\nabla(u_1-u_2)\,\varphi\,\mathrm{d}x\right| &\leqslant \beta\|\varphi\|_{L^\infty(\Omega)} \\ \times\left(\int_{\{0<u_1-u_2<\varepsilon\}} \big(|\nabla u_1| + |\nabla u_2|\big)^p\,\mathrm{d}x\right)^{1/2}&\left(\frac{1}{\varepsilon^2}\int_{\{0<u_1-u_2<\varepsilon\}} \big(|\nabla u_1| + |\nabla u_2|\big)^{p-2}\big|\nabla(u_1-u_2)\big|^2\,\mathrm{d}x\right)^{1/2},\end{aligned}$$

which tends to zero by (10). This implies (12). □

Remark 1. In Lemma 2.2, we easily pass to the limit in the right-hand side of (12) if f_1 and f_2 belong to $L^{p'}(\Omega)$. In such a case, (12) implies that u_i satisfies

$$-\operatorname{div}\big(a(x, u_i, \nabla u_i)\,\chi_{\{u_2<u_1\}}\big) = f_i\,\chi_{\{u_2<u_1\}} \quad \text{in } \mathcal{D}'(\Omega), \tag{14}$$

$$a(x, u_i, \nabla u_i)\,\chi_{\{u_2<u_1\}}\,n = 0 \quad \text{on } \partial\Omega, \tag{15}$$

where n is the exterior normal vector to $\partial\Omega$. Since $f_i\,\chi_{\{u_2<u_1\}} \in L^{p'}(\Omega)$, assertion (15) is not only formal but takes place in $X' = W^{-(1-\frac{1}{p}),p'}(\partial\Omega)$, the dual space of $X = W^{1-\frac{1}{p},p}(\partial\Omega)$, when the boundary $\partial\Omega$ is sufficiently smooth for the functions of $W^{1,p}(\Omega)$ to have traces in $W^{1-\frac{1}{p},p}(\partial\Omega)$. Note that for a general boundary $\partial\Omega$ without any type of regularity, one can define the space of traces as $X = W^{1,p}(\Omega)/W_0^{1,p}(\Omega)$; assertion (15) then takes places in X'.

Remark 2. Property (14) has been proved in [3] for a perturbation of problem (1) by a nonlinear lower order term with natural growth.

Remark 3. When the function a is assumed to satisfy assumptions which are more general than (7) and (8), property (15) is no more true in general, but one can still prove (see [6–8]) that two solutions of (1) satisfy $(a(x, u_1, \nabla u_1) - a(x, u_2, \nabla u_2))\,n = 0$ on $\partial\Omega$.

2.1. Proof of Theorem 1.1

From Lemma 2.2 and $f_1 \leqslant f_2$, we deduce that

$$\int_{\{u_2<u_1\}} \left(a\left(x, u_2, \frac{\mathrm{d}u_2}{\mathrm{d}x}\right) - a\left(x, u_1, \frac{\mathrm{d}u_1}{\mathrm{d}x}\right)\right)\frac{\mathrm{d}\varphi}{\mathrm{d}x}\,\mathrm{d}x \geqslant 0, \tag{16}$$

for every $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geqslant 0$ a.e. in Ω. Replacing φ by $\varphi + \|\varphi\|_{L^\infty(\Omega)}$, we deduce that (16) holds for every $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, thus by density for every $\varphi \in W^{1,p}(\Omega)$, and (16) is therefore an equality and not only an inequality. Since in the one-dimensional case every function of $L^p(\Omega)$ is the derivative of a function of $W^{1,p}(\Omega)$, we get

$$\left(a\left(x, u_2, \frac{\mathrm{d}u_2}{\mathrm{d}x}\right) - a\left(x, u_1, \frac{\mathrm{d}u_1}{\mathrm{d}x}\right)\right)\chi_{\{u_2<u_1\}} = 0 \quad \text{a.e. in } \Omega. \tag{17}$$

Adding and subtracting $a(x, u_2, \frac{\mathrm{d}u_1}{\mathrm{d}x})$ in (17), then multiplying by $\frac{\mathrm{d}(u_2-u_1)}{\mathrm{d}x}$ and using (6), (7) and

$$\left|\frac{\mathrm{d}u_1}{\mathrm{d}x}\right|^{p-1} \leqslant \left(\left|\frac{\mathrm{d}u_1}{\mathrm{d}x}\right| + \left|\frac{\mathrm{d}u_2}{\mathrm{d}x}\right|\right)^{p-1},$$

we obtain

$$\left|\frac{\mathrm{d}(u_2-u_1)}{\mathrm{d}x}\right| \leqslant \frac{\gamma}{\alpha}\left(\left|\frac{\mathrm{d}u_1}{\mathrm{d}x}\right| + \left|\frac{\mathrm{d}u_2}{\mathrm{d}x}\right|\right)|u_2 - u_1|,$$

a.e. in $\{u_2 < u_1\}$. Then, for $\varepsilon > 0$ and T_ε defined by (11), we deduce from Poincaré's inequality in $W_0^{1,1}(\Omega)$ (recall that $u_1 \leqslant u_2$ on $\partial\Omega$)

$$\begin{aligned}\big|\{u_1 - u_2 > \varepsilon\}\big| &\leqslant \int_\Omega \frac{T_\varepsilon(u_1-u_2)^+}{\varepsilon}\,\mathrm{d}x \leqslant C\int_\Omega \left|\frac{\mathrm{d}}{\mathrm{d}x}\frac{T_\varepsilon(u_1-u_2)^+}{\varepsilon}\right|\mathrm{d}x \\ &= \frac{C}{\varepsilon}\int_{\{0<u_1-u_2<\varepsilon\}} \left|\frac{\mathrm{d}(u_1-u_2)}{\mathrm{d}x}\right|\mathrm{d}x \leqslant \frac{C\gamma}{\alpha}\int_{\{0<u_1-u_2<\varepsilon\}} \left(\left|\frac{\mathrm{d}u_1}{\mathrm{d}x}\right| + \left|\frac{\mathrm{d}u_2}{\mathrm{d}x}\right|\right)\mathrm{d}x.\end{aligned}$$

Taking the limit when ε tends to zero, we deduce $|\{u_1 > u_2\}| = 0$, and then $u_2 \geqslant u_1$ a.e. in Ω. □

2.2. *Proof of Theorem 1.2*

We assume that the sign of f_1 is constant in Ω (the other case is similar). Taking $\varphi = 1$ in (12) for $i = 1$, we deduce that $\langle f_1, T_\varepsilon(u_1 - u_2)^+/\varepsilon\rangle$ tends to zero when ε tends to zero. But when the sign of $f_1 \in W^{-1,p'}(\Omega)$ is constant, it is well known that f_1 can be identified with a nonnegative or nonpositive Radon measure which vanishes on the sets of zero $W_0^{1,p}$-capacity. Therefore one has $f_1\,\chi_{\{u_2<u_1\}} = 0$ and the right-hand side of (12) is zero for $i = 1$. Taking then $\varphi = T_k(u_1)$ in (12) for $i = 1$, and passing to the limit when k tends to infinity, we get $\int_{\{u_2<u_1\}} a(x, u_1, \nabla u_1)\nabla u_1\,\mathrm{d}x = 0$, which from (6) and $a(x, s, 0) = 0$ (which results from (8)) implies that $\nabla u_1 = 0$ a.e. in $\{u_2 < u_1\}$. By (9) this also implies $\nabla u_2 = 0$ a.e. in $\{0 < u_1 - u_2 < \varepsilon\}$ for every $\varepsilon > 0$, then (using large values of ε) in $\{u_2 < u_1\}$. Since $u_1 \leqslant u_2$ a.e. on $\partial\Omega$, Poincaré's inequality yields

$$\int_\Omega \big|(u_1-u_2)^+\big|^p\,\mathrm{d}x \leqslant C\int_{\{u_2<u_1\}} \big|\nabla(u_1-u_2)\big|^p\,\mathrm{d}x = 0,$$

which implies $u_1 \leqslant u_2$ a.e. in Ω. □

3. The case where Ω is unbounded

When Ω is unbounded, the results given in the present Note continue to hold (with the same proofs) in a convenient setting where the space $W_0^{1,p}(\Omega)$ is replaced by an adequate space.

A first setting is as follows: if $1 < p < N$, and if Ω is an arbitrary open set, we consider the space $D_0^{1,p}(\Omega)$ obtained by completion of $\mathcal{D}(\Omega)$ for the norm $\|u\|_{D_0^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)}$. Then in (1) we replace $W_0^{1,p}(\Omega)$ by $D_0^{1,p}(\Omega)$, and we take f in $(D_0^{1,p}(\Omega))'$ and U_0 such that $\varphi U_0 \in W^{1,p}(\Omega)$, for every $\varphi \in \mathcal{D}(\mathbb{R}^N)$, and $\nabla U_0 \in L^p(\Omega)$. This formulation corresponds to a Dirichlet boundary condition both on $\partial\Omega$ and at infinity.

A second setting is as follows: if $1 < p < +\infty$, we assume that there exists a ball B_R such that $\mathrm{Cap}_{1,p}(\Omega^c \cap B_R; B_{2R}) > 0$. We define the space $V_0^{1,p}(\Omega) = \{u\colon \varphi u \in W_0^{1,p}(\Omega),\ \forall \varphi \in \mathcal{D}(\mathbb{R}^N),\ \nabla u \in L^p(\Omega)\}$, endowed with the norm $\|u\|_{V_0^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)}$. Then in (1) we replace $W_0^{1,p}(\Omega)$ by $V_0^{1,p}(\Omega)$, and we take f in $(V_0^{1,p}(\Omega))'$ and U_0 as above. This formulation corresponds to a Dirichlet boundary condition on $\partial\Omega$ and to a Neumann boundary condition at infinity. A generalization of this setting to consider the Dirichlet boundary condition on a part of $\partial\Omega$ and the Neumann boundary condition on its complementary, by a convenient modification of the definition of the space $V_0^{1,p}(\Omega)$.

In both settings the above proofs remain unchanged since one can still use Poincaré's inequality in $\Omega \cap B_S$ (with S large enough) for functions in $V_0^{1,p}(\Omega)$, or Sobolev's inequality for functions in $D_0^{1,p}(\Omega)$.

In particular this proves the uniqueness of the capacitary potential (since in that case $f = 0$) for operators which satisfy (6), (7) and (8).

Acknowledgements

The research of J. Casado-Díaz was partially funded by the Spanish Ministerio de Ciencia y Tecnología Grant BFM2002-00672.

References

[1] M. Artola, Sur une classe de problèmes paraboliques quasilinéaires, Boll. Un. Mat. Ital. 5 (1986) 51–70.
[2] L. Boccardo, T. Gallouët, F. Murat, Unicité de la solution de certaines équations elliptiques non linéaires, C. R. Acad. Sci. Paris 315 (I) (1992) 1159–1164.
[3] L. Boccardo, F. Murat, J.-P. Puel, Quelques propriétés des opérateurs elliptiques quasi linéaires, C. R. Acad. Sci. Paris 307 (I) (1988) 749–752.
[4] J. Carrillo, Unicité des solutions du type Kruskov pour des problèmes elliptiques avec des termes de transport non linéaires, C. R. Acad. Sci. Paris 303 (I) (1986) 189–192.
[5] J. Carrillo, M. Chipot, On some elliptic equations involving derivatives of the nonlinearity, Proc. Roy. Soc. Edinburgh Sect. A 100 (1985) 281–294.
[6] J. Casado-Díaz, Minimal and maximal solutions for Dirichlet pseudomonotone problems, Nonlinear Anal. 43 (2001) 277–291.
[7] J. Casado-Díaz, The capacity for pseudomonotone operators, Potential Anal. 14 (2001) 73–91.
[8] J. Casado-Díaz, A. Porretta, Existence and comparison of maximal and minimal solutions for pseudomonotone elliptic problems in L^1, Nonlinear Anal. 53 (2003) 351–373.
[9] M. Chipot, G. Michaille, Uniqueness results and monotonicity properties for strongly nonlinear elliptic variational inequalities, Ann. Sc. Norm. Sup. Pisa 16 (1989) 137–166.
[10] J. Leray, J.-L. Lions, Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty–Browder, Bull. Soc. Math. France 93 (1965) 97–107.
[11] J.-L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris, 1969.
[12] A. Mokrane, F. Murat, The Lewy–Stampacchia inequality for bilateral problems, Ricerche Mat. 53 (2004) 139–182.
[13] A. Porretta, Uniqueness of solutions for some nonlinear Dirichlet problems, Nonlinear Differential Equations Appl. (NoDEA) 11 (2004) 407–430.