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Algebraic Geometry

Construction of Galois covers of curves with groups of SL_2-type ☆

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Abstract

We give a construction of étale Galois covers of algebraic curves over a field of positive characteristic with a prescribed system of finite groups of SL_2-type. ***To cite this article: C.-F. Yu, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Construction de rêvetements galoisiens avec groupes d'un SL_2-type. On donne une construction de rêvetements Galoisiens étales de courbes algébriques définies sur un corps de caractéristique positive avec un système prescrit de groupes finis d'un SL_2-type. ***Pour citer cet article : C.-F. Yu, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

In this Note we give a construction of étale Galois covers of algebraic curves over a field of positive characteristic with a prescribed system of finite groups. Consider a datum (p, ℓ, R) as defined follows:

- p and ℓ are different rational primes;
- R is the ring of integers of a finite product L of local fields over $\mathbb{Q}_\ell$.

The question studied here is the following:

> **(Q)** Can one find a smooth connected projective algebraic curve X over $\overline{\mathbb{F}}_p$ so that for any positive integer m there is a connected étale Galois cover $\pi_m : Y_m \to X$ with Galois group $G_m = \mathrm{SL}_2(R/\ell^m R)$? Furthermore, can one make the covers $\pi_m : Y_m \to X$ compatible with the projective system (G_m)?

We answer the question **(Q)** affirmatively, namely we prove the following:

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Theorem 1.1. *Given a datum (p,ℓ,R) as above, then there is a smooth connected projective curve X over $\overline{\mathbb{F}}_p$ and a compatible system of connected étale Galois covers $\pi_m : Y_m \to X$ with Galois group $\mathrm{SL}_2(R/\ell^m R)$.*

We find a totally real number field F of degree $d = \dim_{\mathbb{Q}_\ell} L$ so that (i) $O_F \otimes \mathbb{Z}_\ell \simeq R$ and (ii) the prime p splits completely in F. Let $\mathbf{M}_F$ be the Hilbert modular variety associated to the totally real field F. The curve X is constructed in the reduction $\mathbf{M}_F \otimes \overline{\mathbb{F}}_p$ modulo p by vanishing $d-1$ Hasse invariants. The cover Y_m arises from the monodromy group for the ℓ^m-torsion subgroup of the universal family restricted on X.

The main tool is the ℓ-adic monodromy of Hecke invariant subvarieties in the moduli spaces of Abelian varieties developed by Chai [1]. This technique confirms that the curves X and Y_m constructed as above are irreducible. The main theorem for Hilbert modular varieties is stated in Section 2.

The construction above provides a solution to the question **(Q)** when $d > 1$. In case of $d = 1$, one replaces R by $\mathbb{Z}_\ell \times \mathbb{Z}_\ell$ and proceeds the same construction. By replacing the covers Y_m by $Y_m/(1 \times \mathrm{SL}_2(\mathbb{Z}/\ell^m\mathbb{Z}))$, one yields a desired compatible system of étale Galois covers.

2. Hecke invariant subvarieties

In this section we describe a theorem of Chai on Hecke invariant subvarieties in a Hilbert modular variety.

Let F be a totally real number field of degree g and O_F be the ring of integers in F. Let V be a 2-dimensional vector space over F and $\psi : V \times V \to \mathbb{Q}$ be a $\mathbb{Q}$-bilinear non-degenerate alternating form such that $\psi(ax,y) = \psi(x,ay)$ for all $x, y \in V$ and $a \in F$. We choose and fix a self-dual O_F-lattice $V_\mathbb{Z} \subset V$. Let p be a fixed rational prime, not necessarily unramified in F. We choose a projective system of primitive prime-to-pth roots of unity $\zeta = (\zeta_m)_{(m,p)=1} \subset \overline{\mathbb{Q}} \subset \mathbb{C}$. We also fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. For any prime-to-$p$ integer $m \geqslant 1$ and any connected $\mathbb{Z}_{(p)}[\zeta_m]$-scheme S, we obtain an isomorphism $\zeta_m : \mathbb{Z}/m\mathbb{Z} \xrightarrow{\sim} \mu_m(S)$.

Let $n \geqslant 3$ be a prime-to-p positive integer and ℓ is a prime with $(\ell, pn) = 1$. Let $m \geqslant 0$ be a non-negative integer. Denote by $\mathbf{M}_{F,n\ell^m}$ the moduli space over $\mathbb{Z}_{(p)}[\zeta_{n\ell^m}]$ that parametrizes equivalence classes of objects $(A,\lambda,\iota,\eta)_S$ over a connected locally Noetherian $\mathbb{Z}_{(p)}[\zeta_{n\ell^m}]$-scheme S, where

- (A,λ) is a principally polarized Abelian scheme over S of relative dimension g;
- $\iota : O_F \to \mathrm{End}_S(A)$ is a ring monomorphism such that $\lambda \circ \iota(a) = \iota(a)^t \circ \lambda$ for all $a \in O_F$; and
- $\eta : V_\mathbb{Z}/n\ell^m V_\mathbb{Z} \xrightarrow{\sim} A[n\ell^m](S)$ is an O_F-linear isomorphism such that $e_\lambda(\eta(x),\eta(y)) = \zeta_{n\ell^m}(\psi(x,y))$, for $x, y \in V_\mathbb{Z}/n\ell^m V_\mathbb{Z}$, where e_λ is the Weil pairing induced by the polarization λ.

Let G be the automorphism group scheme over $\mathbb{Z}$ associated to the pair $(V_\mathbb{Z}, \psi)$. Let $\Gamma(n\ell^m) \subset G(\mathbb{Z})$ be the kernel of the reduction map $G(\mathbb{Z}) \to G(\mathbb{Z}/n\ell^m\mathbb{Z})$. It is well-known that one has the complex uniformization $\mathbf{M}_{F,n\ell^m}(\mathbb{C}) \simeq \Gamma(n\ell^m)\backslash G(\mathbb{R})/\mathrm{SO}(2,\mathbb{R})^g$. In particular, the geometric generic fiber $\mathbf{M}_{F,n\ell^m} \otimes \overline{\mathbb{Q}}$ is connected. It follows from the arithmetic compactification constructed in Rapoport [5] that the geometric special fiber $\mathbf{M}_{F,n\ell^m} \otimes \overline{\mathbb{F}}_p$ is also connected. Write $M_{n\ell^m} := \mathbf{M}_{F,n\ell^m} \otimes \overline{\mathbb{F}}_p$ for the reduction modulo p of the moduli scheme $\mathbf{M}_{F,n\ell^m}$. We have a natural morphism $\pi_{m,m'} : M_{n\ell^{m'}} \to M_{n\ell^m}$, for $m < m'$, which is induced from the map $(A,\lambda,\iota,\eta) \mapsto (A,\lambda,\iota,\ell^{m'-m}\eta)$. Let $\widetilde{M}_n := (M_{n\ell^m})_{m \geqslant 0}$ be the tower of this projective system.

Let $(\mathcal{X},\lambda,\iota,\eta) \to M_n$ be the universal family. The cover $M_{n\ell^m}$ represents the étale sheaf

$$\mathcal{P}_m := \underline{Isom}_{M_n}\big((V_\mathbb{Z}/\ell^m V_\mathbb{Z}, \psi), (\mathcal{X}[\ell^m], e_\lambda); \zeta_{\ell^m}\big) \tag{1}$$

of O_F-linear symplectic level-ℓ^m structures with respect to ζ_{ℓ^m}. This is a $G(\mathbb{Z}/\ell^m\mathbb{Z})$-torsor. Let $\bar{x}$ be a geometric point in M_n. Choose an O_F-linear isomorphism $y : V \otimes \mathbb{Z}_\ell \simeq T_\ell(\mathcal{X}_{\bar{x}})$ which is compatible with the polarizations with respect to ζ. This amounts to choose a geometric point in $\widetilde{M}_n$ over the point $\bar{x}$. The action of the geometric fundamental group $\pi_1(M_n,\bar{x})$ on the system of fibers $(\mathcal{X}_{\bar{x}}[\ell^m])_m$ gives rise to the monodromy representation $\rho_{M_n,\ell} : \pi_1(M_n,\bar{x}) \to \mathrm{Aut}_{O_F}(T_\ell(\mathcal{X}_{\bar{x}}), e_\lambda)$, and through the choice of y to the monodromy representation (using the same notation)

$$\rho_{M_n,\ell} : \pi_1(M_n,\bar{x}) \to G(\mathbb{Z}_\ell). \tag{2}$$

The connectedness of $\widetilde{M}_n$ affirms that the monodromy map $\rho_{M_n,\ell}$ is surjective.

For any non-negative integer $m \geqslant 0$, let $\mathcal{H}_{\ell,m}$ be the moduli space over $\overline{\mathbb{F}}_p$ that parametrizes equivalence classes of objects $(\underline{A}_i = (A_i,\lambda_i,\iota_i,\eta_i), i = 1,2,3; \varphi_1,\varphi_2)$ as the diagram $\underline{A}_1 \xleftarrow{\varphi_1} \underline{A}_3 \xrightarrow{\varphi_2} \underline{A}_2$, where

– each $\underline{A}_i$ is a g-dimensional polarized Abelian O_F-variety with a symplectic level-n structure, and both $\underline{A}_1$ and $\underline{A}_2$ are in M_n;
– φ_1 and φ_2 are O_F-linear isogenies of degree ℓ^m that preserve the polarizations and level structures.

Let $\mathcal{H}_\ell := \bigcup_{m\geqslant 0}\mathcal{H}_{\ell,m}$. An ℓ-adic Hecke correspondence is an irreducible component $\mathcal{H}$ of $\mathcal{H}_\ell$ together with natural projections pr_1 and pr_2. A subset Z of M_n is called *ℓ-adic Hecke invariant* if $\mathrm{pr}_2(\mathrm{pr}_1^{-1}(Z)) \subset Z$ for any ℓ-adic Hecke correspondence $(\mathcal{H}, \mathrm{pr}_1, \mathrm{pr}_2)$. If Z is an ℓ-adic Hecke invariant, locally closed subvariety of M_n, then the Hecke correspondences induce correspondences on the set $\Pi_0(Z)$ of geometrically irreducible components. We say $\Pi_0(Z)$ is *ℓ-adic Hecke transitive* if the ℓ-adic Hecke correspondences operate transitively on $\Pi_0(Z)$, that is, for any two maximal points η_1, η_2 of Z there is an ℓ-Hecke correspondence $(\mathcal{H}, \mathrm{pr}_1, \mathrm{pr}_2)$ so that $\eta_2 \in \mathrm{pr}_2(\mathrm{pr}_1^{-1}(\eta_1))$.

Theorem 2.1. (Chai) *Let Z be an ℓ-adic Hecke invariant, smooth locally closed subvariety of M_n. Let $\bar\eta$ be a geometric generic point of an irreducible component Z^0 of Z. Suppose that the Abelian variety $A_{\bar\eta}$ corresponding to the point $\bar\eta$ is not supersingular, and that the set $\Pi_0(Z)$ is ℓ-adic Hecke transitive. Then the monodromy representation $\rho_{Z^0,\ell}:\pi_1(Z^0,\bar\eta)\to G(\mathbb{Z}_\ell)$ is surjective and Z is irreducible.*

The proof of this theorem is given by Chai [1] for Siegel modular varieties, which uses the semi-simplicity of the geometric monodromy group of a pure $\mathbb{Q}_\ell$-sheaf on a variety over a finite field due to Grothendieck and Deligne [2, Corollary 1.3.9 and Theorem 3.4.1]. Chai's proof also works for Hilbert modular varieties as stated in Theorem 2.1; see an expository account in [7, Section 6]. Let $Z_m := M_{n\ell^m}\times_{M_n} Z$. Theorem 2.1 also implies that Z_m is irreducible provided the conditions for Z are satisfied.

3. The construction

Lemma 3.1 *(Krasner's Lemma). Let k be a local field of characteristic zero and $f(X)$ be a monic separable polynomial of degree n. If $g(X)$ is a monic polynomial of degree n whose coefficients are sufficiently close to those of $f(X)$. Then $g(X)$ is separable and there is an isomorphism of k-algebras $k[X]/(g(X))\simeq k[X]/(f(X))$.*

Lemma 3.2. *Let S be a finite set of places of a number field k. Let L_v, for each $v\in S$, be a product of local fields over k_v of same degree $[L_v : k_v] = n$, where k_v is the completion of k at v. Then there is a number field F over k of degree n such that $F\otimes_k k_v\simeq L_v$ for all $v\in S$.*

Proof. This follows from an effective version of Hilbert's irreducibility theorem [3, Theorem 1.3] and Krasner's lemma (Lemma 3.1). □

Corollary 3.3. *Given a datum (p,ℓ,R) as before, there is a totally real number field F of degree $d=\dim_{\mathbb{Q}_\ell} L$ so that* (i) *$O_F\otimes\mathbb{Z}_\ell\simeq R$ and* (ii) *the prime p splits completely in F.*

Proof. This follows from Lemma 3.2. □

Assume that $d>1$. Let F be a totally real number field as in Corollary 3.3. Write the set of ring homomorphisms from O_F to $\mathbb{F}_p$ as $\{\sigma_1,\dots,\sigma_d\}$. Define modular varieties M_n and $M_{n\ell^m}$ over $\overline{\mathbb{F}}_p$ as in Section 2 (with a choice of a system of roots of unity ζ). Let $a:(\mathcal{X},\lambda,\iota,\eta)\to M_n$ be the universal family. Let $H^1_{\mathrm{DR}}(\mathcal{X}/M_n)$ be the algebraic de Rham cohomology; it has a decomposition

$$H^1_{\mathrm{DR}}(\mathcal{X}/M_n)=\bigoplus_{i=1}^{d} H^1_{\mathrm{DR}}(\mathcal{X}/M_n)^i \tag{3}$$

with respect to the O_F-action, where $H^1_{\mathrm{DR}}(\mathcal{X}/M_n)^i$ is the σ_i-isotypic component. Each component $H^1_{\mathrm{DR}}(\mathcal{X}/M_n)^i$ is a locally free $\mathcal{O}_{M_n}$-module of rank 2. The Hodge filtration

$$0\to\omega_{\mathcal{X}/M_n}\to H^1_{\mathrm{DR}}(\mathcal{X}/M_n)\to R^1a_*\mathcal{O}_{\mathcal{X}}\to 0 \tag{4}$$

also has the same decomposition

$$0 \to \omega^i_{\mathcal{X}/M_n} \to H^1_{\mathrm{DR}}(\mathcal{X}/M_n)^i \to R^1 a_* \mathcal{O}^i_{\mathcal{X}} \to 0, \tag{5}$$

for all $1 \leqslant i \leqslant d$. Let $F_{\mathcal{X}/M_n} : \mathcal{X} \to \mathcal{X}^{(p)}$ be the relative Frobenius morphism, where $\mathcal{X}^{(p)}$ is base change of $\mathcal{X}$ by the absolute Frobenius morphism $F_{M_n} : M_n \to M_n$. The morphism $F_{\mathcal{X}/M_n}$, by functoriality, induces an $\mathcal{O}_{M_n}$-linear map $F_i : R^1 a_* \mathcal{O}^i_{\mathcal{X}^{(p)}} \to R^1 a_* \mathcal{O}^i_{\mathcal{X}}$. By duality, one has $h_i := F_i^\vee : \omega^i_{\mathcal{X}/M_n} \to \omega^i_{\mathcal{X}^{(p)}/M_n}$. Since $\omega^i_{\mathcal{X}^{(p)}/M_n} \simeq (\omega^i_{\mathcal{X}/M_n})^{\otimes p}$, the homomorphism h_i is an element in $H^0(M_n, \mathcal{L}^i)$, where $\mathcal{L}^i := (\omega^i_{\mathcal{X}/M_n})^{\otimes(p-1)}$.

Let X be the closed subscheme of M_n defined by $h_i = 0$ for $2 \leqslant i \leqslant d$. Let $Y_m := M_{n\ell^m} \times_{M_n} X$. It is clear that X is stable under all ℓ-adic Hecke correspondences. We verify the conditions in Theorem 2.1:

Lemma 3.4.

(1) *The subscheme X is a smooth projective curve over $\overline{\mathbb{F}}_p$.*
(2) *Any maximal point of X is not supersingular.*
(3) *The set $\Pi_0(X)$ of irreducible components is ℓ-adic Hecke transitive.*

Proof. Since points in X are not ordinary, it follows from the semi-stable reduction theorem that X is proper. By the Serre–Tate theorem, the deformations in M_n are the same as a product of deformations in an elliptic modular curve. It is well-known that the zero locus of the Hasse invariant is reduced and ordinary elliptic curves are dense in the mod p of an elliptic modular curve. From this the statements (1) and (2) follows. The statement (3) is a special case of [7, Theorem 5.1]. □

By Theorem 2.1, the curves X and Y_m are irreducible. One also has $\mathrm{Aut}(Y_m/X) = G(\mathbb{Z}/\ell^m\mathbb{Z}) = \mathrm{SL}_2(R/\ell^m R)$. The construction is complete. This finishes the proof of Theorem 1.1. The following question to which we do not know the answer. *What is the genus of X as above?*

Remark 1. (i) Regardless the construction, the statement itself of Theorem 1.1 can be proved by standard methods. Such an X can be obtained by a specialization argument as in [6, Prop. 2.5 and Cor. 3.5] and the Grothendieck specialization theorem for algebraic fundamental groups.

(ii) Consider quaternion algebras B over a totally real number field F so that B splits at exactly one of real places of F and B splits at all primes of F over p. Let $\mathbf{M}_B$ be the Shimura curve associated to B and take $X := \mathbf{M}_B \otimes \overline{\mathbb{F}}_p$ to be the reduction modulo p (see [4] for a nice summary of Ihara's work on Shimura curves). This exhibits a solution to the question **(Q)** for not just a prescribed system arising from SL_2 over $F \otimes \mathbb{Q}_\ell$ but also that from its inner twist.

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