Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 345 (2007) 229–232
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Probability Theory

The characteristic polynomial on compact groups with Haar measure: some equalities in law

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Received 18 June 2007; accepted 21 June 2007

Available online 31 July 2007

Presented by Marc Yor

Abstract

This Note presents some equalities in law for $Z_N := \det(\mathrm{Id} - G)$, where G is an element of a subgroup of the set of unitary matrices of size N, endowed with its unique probability Haar measure. Indeed, under some general conditions, Z_N can be decomposed as a product of independent random variables, whose laws are explicitly known. Our results can be obtained in two ways: either by a recursive decomposition of the Haar measure (Section 1) or by previous results by Killip and Nenciu (2004) on orthogonal polynomials with respect to some measure on the unit circle (Section 2). This latter method leads naturally to a study of determinants of a class of principal submatrices (Section 3). ***To cite this article: P. Bourgade et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Quelques égalités en lois pour le polynôme caractéristique sur des groupes compacts, sous la mesure de Haar. Cette Note présente quelques égalités en loi pour $Z_N := \det(\mathrm{Id} - G)$, où G est un sous-groupe de l'ensemble des matrices unitaires de taille N, muni de son unique mesure de Haar normalisée. En effet, sous des conditions assez générales, Z_N peut être décomposé comme le produit de variables aléatoires indépendantes, dont on connait la loi explicitement. Notre résultat peut être obtenu de deux manières : soit par une décomposition récursive de la mesure de Haar (Partie 1) soit en utilisant un résultat de Killip et Nenciu (2004) à propos des polynômes orthogonaux relativement à une certaine mesure sur le cercle unité (Partie 2). Cette dernière méthode nous conduit naturellement à l'étude des déterminants de certaines sous-matrices (Partie 3). ***Pour citer cet article : P. Bourgade et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

In this Note, $\langle a, b\rangle$ denotes the Hermitian product of two elements a and b in $\mathbb{C}^N$ (the dimension is implicit).

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2007.06.023

1. A recursive decomposition, consequences

1.1. The general equality in law

Let $\mathcal{G}$ be a subgroup of $U(N)$, the group of unitary matrices of size N. Let $(e_1, \dots, e_N)$ be an orthonormal basis of $\mathbb{C}^N$ and $\mathcal{H} := \{H \in \mathcal{G} \mid H(e_1) = e_1\}$, the subgroup of $\mathcal{G}$ which stabilizes e_1. For a generic compact group $\mathcal{A}$, we write $\mu_{\mathcal{A}}$ for the unique Haar probability measure on $\mathcal{A}$. Then we have the following theorem:

Theorem 1.1. *Let M and H be independent matrices, $M \in \mathcal{G}$ and $H \in \mathcal{H}$ with distribution $\mu_{\mathcal{H}}$. Then $MH \sim \mu_{\mathcal{G}}$ if and only if $M(e_1) \sim f(\mu_{\mathcal{G}})$, where f is the map $f : G \mapsto G(e_1)$.*

Let $\mathcal{M}$ be the set of elements of $\mathcal{G}$ which are reflections with respect to a hyperplane of $\mathbb{C}^N$. Define also

$$g : \begin{cases} \mathcal{H} \to U(N-1), \\ H \mapsto H_{\mathrm{span}(e_2,\dots,e_N)}, \end{cases}$$

where $H_{\mathrm{span}(e_2,\dots,e_N)}$ is the restriction of H to $\mathrm{span}(e_2, \dots, e_N)$. Now suppose that $\{G(e) \mid G \in \mathcal{G}\} = \{M(e) \mid M \in \mathcal{M}\}$. Under this additional condition the following Theorem can be proven, using Theorem 1.1 and elementary manipulations of determinants:

Theorem 1.2. *Let $G \sim \mu_{\mathcal{G}}$, $G' \sim \mu_{\mathcal{G}}$ and $H \sim g(\mu_{\mathcal{H}})$ be independent. Then*

$$\det(\mathrm{Id}_N - G) \overset{\mathrm{law}}{=} \big(1 - \langle e_1, G'(e_1)\rangle\big) \det(\mathrm{Id}_{N-1} - H).$$

1.2. Examples: the unitary group, the group of permutations

Take $G = U(N)$. As all reflections with respect to a hyperplane of $\mathbb{C}^N$ are elements of G, iterations of Theorem 1.2 lead to the following corollary.

Corollary 1.3. ([1]) *Let $G \in U(N)$ be $\mu_{U(N)}$ distributed. Then*

$$\det(\mathrm{Id}_N - G) \overset{\mathrm{law}}{=} \prod_{k=1}^{N} \big(1 - \mathrm{e}^{\mathrm{i}\theta_k} \sqrt{\beta_{1,k-1}}\big),$$

with $\theta_1, \dots, \theta_N, \beta_{1,0}, \dots, \beta_{1,N-1}$ independent random variables, the θ_k's uniformly distributed on $(0, 2\pi)$ and the $\beta_{1,j}$'s $(0 \leqslant j \leqslant N-1)$ being beta distributed with parameters 1 and j (by convention, $\beta_{1,0}$ is the Dirac distribution at 1).

The group $\mathcal{S}_N$ of permutations of size N gives another possible application. Identify an element $\sigma \in \mathcal{S}_N$ with the matrix $(\delta^j_{\sigma(i)})_{1 \leqslant i,j \leqslant N}$ (δ is Kronecker's symbol). As $\det(\mathrm{Id}_N - \sigma)$ is equal to 0, we prefer to deal with the group $\tilde{\mathcal{S}}_N$ of matrices $(\mathrm{e}^{\mathrm{i}\theta_j} \delta^j_{\sigma(i)})_{1 \leqslant i,j \leqslant N}$, with $\sigma \in \mathcal{S}_N$ and $\theta_1, \dots, \theta_N$ independent uniform random variables on $(0, 2\pi)$. As previously, iterations of Theorem 1.2 give the following result:

Corollary 1.4. *Let $S_N \in \tilde{\mathcal{S}}_N$ be $\mu_{\tilde{\mathcal{S}}_N}$ distributed. Then*

$$\det(\mathrm{Id}_N - S_N) \overset{\mathrm{law}}{=} \prod_{k=1}^{N} \big(1 - \mathrm{e}^{\mathrm{i}\theta_k} X_k\big),$$

with $\theta_1, \dots, \theta_N, X_1, \dots, X_N$ independent random variables, the θ_k's uniformly distributed on $(0, 2\pi)$ and the X_k's Bernoulli variables: $\mathbb{P}(X_k = 1) = 1/k$, $\mathbb{P}(X_k = 0) = 1 - 1/k$.

2. Characteristic polynomials as orthogonal polynomials

We now show how Corollary 1.3 can be obtained as a consequence of a result by Killip and Nenciu [2].

2.1. A result by Killip and Nenciu

Let $\mathbb{D}$ be the open unit disk $\{z \in \mathbb{C}\colon |z| < 1\}$ and $\partial\mathbb{D}$ the unit circle. Let $(e_1, \ldots, e_N)$ be the canonical basis of $\mathbb{C}^N$. If $G \in U(N)$, and if e_1 is cyclic for G, the spectral measure for the pair (G, e_1) is the unique probability ν on $\partial\mathbb{D}$ such that, for every integer $k \geqslant 0$

$$\langle e_1, G^k e_1 \rangle = \int_{\partial\mathbb{D}} z^k \,\mathrm{d}\nu(z). \tag{1}$$

In fact, we have the expression

$$\nu = \sum_{j=1}^{N} \pi_j \delta_{\mathrm{e}^{\mathrm{i}\zeta_j}}$$

where $(e^{i\zeta_j}, j = 1, \ldots, N)$ are the eigenvalues of G and where $\pi_j = |\langle e_1, \Pi e_j \rangle|^2$ with Π a unitary matrix diagonalizing G.

The relation (1) allows to define an isometry from $\mathbb{C}^N$ equipped with the basis $(e_1, Ge_1, \ldots, G^{N-1}e_1)$ into the subspace of $L^2(\partial\mathbb{D}; \mathrm{d}\nu)$ spanned by the family $(1, z, \ldots, z^N)$. The endomorphism G is then a representation of the multiplication by z.

From the linearly independent family of monomials $\{1, z, z^2, \ldots, z^{N-1}\}$ in $L^2(\partial\mathbb{D}, \nu)$, we construct an orthogonal basis $\Phi_0, \ldots, \Phi_{N-1}$ of monic polynomials by the Gram-Schmidt procedure. The Nth degree polynomial obtained this way is

$$\Phi_N(z) = \prod_{j=1}^{N} \left(z - \mathrm{e}^{\mathrm{i}\zeta_j}\right),$$

i.e. the characteristic polynomial of G. The Φ_k's $(k = 0, \ldots, N)$ obey the Szegö recursion relation:

$$\Phi_{j+1}(z) = z\Phi_j(z) - \bar{\alpha}_j \Phi_j^*(z) \tag{2}$$

where $\Phi_j^*(z) = z^j\, \overline{\Phi_j(\bar{z}^{-1})}$. The coefficients α_j's $(j \geqslant 0)$ are called Schur or Verblunsky coefficients and satisfy the condition $\alpha_0, \ldots, \alpha_{N-2} \in \mathbb{D}$ and $\alpha_{N-1} \in \partial\mathbb{D}$. There is a bijection between this set of coefficients and the set of spectral probability measures ν (Verblunsky's theorem). If $G \sim \mu_{U(N)}$, then we know the exact distribution of the Verblunsky coefficients:

Theorem 2.1. (*Killip and Nenciu [2]*) *Let $G \in U(N)$ be $\mu_{U(N)}$ distributed. The Verblunsky parameters $\alpha_0, \ldots, \alpha_{N-2}$, α_{N-1} are independent and the density of α_j for $j \leqslant N-1$ is*

$$\frac{N-j-1}{\pi}\left(1 - |z|^2\right)^{N-j-2} 1_{\mathbb{D}}(z)$$

(*for $j = N-1$ by convention this is the uniform measure on the unit circle*).

2.2. Recovering Corollary 1.3

For $z = 1$, Szegö's recursion (2) can be written

$$\Phi_{j+1}(1) = \Phi_j(1) - \overline{\alpha_j}\, \overline{\Phi_j(1)}. \tag{3}$$

Under the Haar measure for G, as α_j is independent of $\Phi_j(1)$ and its distribution is invariant by rotation, (3) easily yields

$$\Phi_{j+1}(1) \stackrel{\text{law}}{=} (1 - \alpha_j)\Phi_j(1).$$

In particular, for $j = N-1$ we get by induction exactly the same result as Corollary 1.3.

Remark. A similar result holds for $\mathrm{SO}(2N)$, and can be shown using either the method of Section 1 or the one in Section 2, with the corresponding result by Killip and Nenciu for the Verblunsky coefficients on the orthogonal group [2].

2.3. Extension

We now consider the whole sequence of polynomials Φ_j, $j \leqslant N$, for $j \leqslant N$ as a sequence of characteristic polynomials. For this purpose, we apply the Gram-Schmidt procedure to $1, z, z^{-1}, z^2, \ldots, z^{p-1}, z^{1-p}, z^p$ if $N = 2p$ and to $1, z, z^{-1}, z^2, \ldots, z^p, z^{-p}$ if $N = 2p+1$ in $L^2(\partial\mathbb{D}); \mathrm{d}\nu)$. In the resulting basis, the mapping $f(z) \mapsto zf(z)$ is represented by a so-called CMV matrix ([2] Appendix B, [4]) denoted by $\mathcal{C}_N(G)$. It is five-diagonal and conjugate to G. For $1 \leqslant j \leqslant N$ let $\mathcal{C}_N^{(j)}(G)$ the principal submatrix of order j of $\mathcal{C}_N(G)$. It is known (see for instance Proposition 3.1 in [4]) that

$$\Phi_j(z) = \det\bigl(z\,\mathrm{Id}_j - \mathcal{C}_N^{(j)}(G)\bigr).$$

From the recursion (3) and looking at the invariance of conditional distributions, we see that

$$\bigl(\det\bigl(\mathrm{Id}_j - \mathcal{C}_N^{(j)}(G)\bigr)\bigr)_{1\leqslant j\leqslant N} = \bigl(\Phi_j(1)\bigr)_{1\leqslant j\leqslant N} \stackrel{\text{law}}{=} \left(\prod_{l=0}^{j}(1-\alpha_l)\right)_{0\leqslant j\leqslant N-1}. \tag{4}$$

It allows a study of the process $(\log \Phi_{\lfloor Nt\rfloor}(1),\ t \in [0,1])$ as a triangular array of (complex) independent random variables. For $t = 1$ the asymptotic behavior is presented in [1] (see (6) below). It is remarkable that for $t < 1$, we do not need any normalization for the CLT.

Theorem 2.2.

(i) *As* $N \to \infty$

$$\bigl(\log\det\bigl(\mathrm{Id}_j - \mathcal{C}_{\lfloor Nt\rfloor}^{(j)}(G)\bigr);\ t \in [0,1)\bigr) \Rightarrow \bigl(\mathbf{B}_{-\frac{1}{2}\log(1-t)};\ t \in [0,1)\bigr), \tag{5}$$

where $\mathbf{B}$ *is a standard complex Brownian motion and* $\Rightarrow$ *stands for the weak convergence of distributions in the set of càdlàg functions on* $[0,1)$*, starting from* 0*, endowed with the Skorokhod topology.*

(ii) *As* $N \to \infty$*,*

$$\frac{\log\det(\mathrm{Id}_N - G)}{\sqrt{2\log N}} \Rightarrow \mathcal{N}_1 + \mathrm{i}\mathcal{N}_2 \tag{6}$$

where $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *are independent standard normal and independent of* $\mathbf{B}$*, and* $\Rightarrow$ *stands for the weak convergence of distributions in* $\mathbb{C}$*.*

This theorem can be proved using the Mellin–Fourier transform of the $1-\alpha_j$'s and independence. This method may also be used to prove large deviations. It is the topic of a companion paper. These results occur in similar way for other random determinants (see [3]).

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