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Algebraic Geometry

A cohomological criterion for semistable parabolic vector bundles on a curve

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Abstract

Let X be an irreducible smooth complex projective curve and $S \subset X$ a finite subset. Fix a positive integer N. We consider all the parabolic vector bundles over X whose parabolic points are contained in S and all the parabolic weights are integral multiples on $1/N$. We construct a parabolic vector bundle V_*, of this type, satisfying the following condition: a parabolic vector bundle E_* of this type is parabolic semistable if and only if there is a parabolic vector bundle F_*, also of this type, such that the underlying vector bundle $(E_* \otimes F_* \otimes V_*)_0$ for the parabolic tensor product $E_* \otimes F_* \otimes V_*$ is cohomologically trivial, which means that $H^i(X, (E_* \otimes F_* \otimes V_*)_0) = 0$ for all i. Given any parabolic semistable vector bundle E_*, the existence of such F_* is proved using a criterion of Faltings which says that a vector bundle E over X is semistable if and only if there is another vector bundle F such that $E \otimes F$ is cohomologically trivial. ***To cite this article: I. Biswas, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Un critère cohomologique pour des fibrés vectoriels paraboliques semistables sur une courbe. Soit X une courbe complexe lisse projective irréductible et $S \subset X$ une partie finie. Fixons un entier positif N. Nous considerons les fibrés vectoriels paraboliques sur X dont les points paraboliques sont contenus dans S et les poids paraboliques sont des multiples entiers de $1/N$. Nous construisons un tel fibré vectoriel parabolique V_*, vérifiant la condition suivante : un fibré vectoriel parabolique E_* du type comme ci-dessus est semistable au sens parabolique si et seulement s'il existe un fibré vectoriel parabolique F_*, aussi de tel type, tel que le fibré vectoriel sous-jacent $(E_* \otimes F_* \otimes V_*)_0$ au produit tensoriel parabolique $E_* \otimes F_* \otimes V_*$ soit cohomologiquement trivial : on a $H^i(X, (E_* \otimes F_* \otimes V_*)_0) = 0$ pour $i = 0, 1$. L'existence d'un tel F_* est démontrée en utilisant un critère de Faltings qui dit qu'un fibré vectoriel E sur X est semistable si et seulement s'il existe un fibré vectoriel F tel que $H^i(X, E \otimes F) = 0$ pour $i = 0, 1$. ***Pour citer cet article : I. Biswas, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. A parabolic vector bundle

Let X be an irreducible smooth projective curve defined over $\mathbb{C}$. Fix a finite subset

$$S = \{p_1, \ldots, p_n\} \subset X. \tag{1}$$

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Also fix a positive integer N. We will consider parabolic vector bundles E_* over X satisfying the following two conditions:

(1) the parabolic points of E_* are contained in the subset S in (1), and
(2) all the parabolic weights of E_* are integral multiples on $1/N$.

(See [5, Section 1] for parabolic bundles.)

There is an algebraic Galois covering

$$f : Y \longrightarrow X \tag{2}$$

satisfying the following conditions:

- the subset of X over which f is ramified contains S, and
- for each points $p_i \in S$,
$$f^{-1}(p_i) = N \cdot f^{-1}(p_i)_{\mathrm{red}}, \tag{3}$$
where $f^{-1}(p_i)_{\mathrm{red}}$ is the reduced inverse image.

(See [6, p. 26, Proposition 1.2.12] for the existence of such a covering f.)

Let

$$\Gamma := \mathrm{Gal}(f)$$

be the Galois group for the covering f. A *Γ-linearized vector bundle* on Y is an algebraic vector bundle E over Y equipped with a lift of the action of Γ as vector bundle automorphisms; this means that the group Γ acts on the total space of E as algebraic automorphisms, and the action of each $\gamma \in \Gamma$ on E is a vector bundle isomorphism of E with $(\gamma^{-1})^*E$.

There is a natural bijective correspondence between the parabolic vector bundles over X of the type mentioned earlier, and the Γ-linearized vector bundles E on Y satisfying the following condition: for each point $z \in f^{-1}(X \setminus S)$, the action on the fiber E_z of the isotropy subgroup for z (for the action of Γ on E) is trivial. See [2] for the details of this bijective correspondence.

Tensor product and direct sum of two parabolic vector bundles can be defined. Similarly, the dual of a parabolic vector bundle is also defined; see [7,3,1]. The earlier mentioned class of parabolic vector bundles is closed under these operations. Furthermore, the above mentioned bijective correspondence between parabolic vector bundles on X and Γ-linearized vector bundles on Y transports the operations of taking tensor product, direct sum and dual of parabolic vector bundles to the operations of taking tensor product, direct sum and dual respectively of Γ-linearized vector bundles. A parabolic vector bundle over X is parabolic semistable if and only if the corresponding Γ-linearized vector bundle on Y is semistable in the usual sense; see [2, p. 318, Lemma 3.13] and [2, p. 308, Lemma 2.7].

A natural parabolic structure on the direct image

$$V = f_*\mathcal{O}_Y \tag{4}$$

on X will be described. For any integer $j \in [1, N]$, let

$$V_j := f_*\mathcal{O}_Y\big(-(N-j)f^{-1}(S)_{\mathrm{red}}\big)$$

be the direct image on X, where N is the integer in (3) and $f^{-1}(S)_{\mathrm{red}}$ is the reduced inverse image of S. Consider the filtration of coherent subsheaves of V

$$V_1 \subset \cdots \subset V_i \subset \cdots \subset V_{N-1} \subset V_N = V.$$

The restriction of this filtration to a point $p_i \in S$ gives a filtration of subspaces

$$0 \subset V_{p_i}^1 \subset \cdots \subset V_{p_i}^j \subset V_{p_i}^{j+1} \subset \cdots \subset V_{p_i}^{N-1} \subset V_{p_i}^N = V_{p_i} \tag{5}$$

of the fiber V_{p_i}; so the subspace $V_{p_i}^j \subset V_{p_i}$ is the image of the fiber $(V_j)_{p_i}$ in V_{p_i}. The dimension of each successive quotient in (5) is $(\#\Gamma)/N$. The parabolic structure on V is defined as follows: The quasiparabolic filtration on each $p_i \in S$ is the one in (5), and the parabolic weight of the subspace $V_{p_i}^j \subset V_{p_i}$ in (5) is $(N-j)/N$.

Let V_* denote the parabolic vector bundle defined by the above parabolic structure on the vector bundle V in (4). We will construct a Γ-linearized vector bundle associated to a parabolic vector bundle related to V_*.

Let $\mathbb{C}(\Gamma)$ denote the group algebra for Γ defined by the formal sums of the form $\sum_{\gamma\in\Gamma} c_\gamma \gamma$ with $c_\gamma \in \mathbb{C}$. Let

$$\widehat{V} := \mathcal{O}_Y \otimes_{\mathbb{C}} \mathbb{C}(\Gamma) \tag{6}$$

be the trivial vector bundle on Y. The natural action of Γ on $\mathbb{C}(\Gamma)$ and the diagonal action of Γ on $\mathcal{O}_Y = Y \times \mathbb{C}$, with Γ acting trivially on $\mathbb{C}$, together define a Γ-linearization on the vector bundle $\widehat{V}$ in (6).

Let V'_* be the parabolic vector bundle on X given by the above Γ-linearized vector bundle $\widehat{V}$; see [2, Section 2c]. The above defined parabolic vector bundle V_* is obtained from V'_* by simply forgetting all the parabolic structures on the complement $X \setminus S$, keeping the underlying vector bundle unchanged. (Note that since f may be ramified over points outside S, the parabolic vector bundle V'_* may have nontrivial parabolic structures outside S.)

2. Criterion for semistability

All parabolic vector bundles will be assumed to satisfy the two conditions stated at the beginning of Section 1.

Theorem 2.1. *A parabolic vector bundle E_* over X is parabolic semistable if and only if there is a parabolic vector bundle F_* such that the vector bundle $(E_* \otimes F_* \otimes V_*)_0$ on X underlying the parabolic tensor product $E_* \otimes F_* \otimes V_*$, where V_* is constructed in Section* 1, *satisfies the following condition*:

$$H^i\big(X, (E_* \otimes F_* \otimes V_*)_0\big) = 0 \tag{7}$$

for all i.

Proof. Let E_* be a parabolic vector bundle over X. First assume that there is a parabolic vector bundle F_* such that (7) holds for all i.

Let $\widehat{E}$ (respectively, $\widehat{F}$) be the unique Γ-linearized vector bundle over the curve Y in (2) corresponding to the parabolic vector bundle E_* (respectively, F_*).

We note that if $\widehat{E}'$ is the Γ-linearized vector bundle over Y corresponding to a parabolic vector bundle E'_* on X, then

$$H^i(Y, \widehat{E}')^\Gamma = H^i(X, E') \tag{8}$$

for all i, where E' is the vector bundle underlying E'_*. Indeed, this follows immediately from the fact that $E' = (f_*\widehat{E}')^\Gamma$ [2, p. 310, (2.9)]. Using (8), and the fact that the correspondence between parabolic vector bundles and Γ-linearized vector bundles is compatible with the tensor product operation, it follows from (7) that

$$H^i(Y, \widehat{E} \otimes \widehat{F} \otimes \widehat{V})^\Gamma = 0 \tag{9}$$

for all i, where $\widehat{V}$ is the vector bundle in (6). Note that since the parabolic vector bundle V_* is obtained from the parabolic vector bundle V'_* associated to the Γ-linearized vector bundle $\widehat{V}$ by forgetting the parabolic structure on $X \setminus S$ keeping the underlying vector bundle unchanged, and both E_* and F_* do not have any parabolic points outside S, the vector bundle underlying the parabolic tensor product $E_* \otimes F_* \otimes V'_*$ is actually identified with the vector bundle $(E_* \otimes F_* \otimes V_*)_0$ underlying the parabolic vector bundle $E_* \otimes F_* \otimes V_*$.

From the definition of $\widehat{V}$ in (6) it follows that

$$H^i(Y, \widehat{E} \otimes \widehat{F}) = \big(H^i(Y, \widehat{E} \otimes \widehat{F}) \otimes_{\mathbb{C}} \mathbb{C}(\Gamma)\big)^\Gamma = H^i(Y, \widehat{E} \otimes \widehat{F} \otimes \widehat{V})^\Gamma. \tag{10}$$

We note that given any finite dimensional complex left Γ-module M, there is a canonical $\mathbb{C}$-linear isomorphism

$$M \to \big(M \otimes_{\mathbb{C}} \mathbb{C}(\Gamma)\big)^\Gamma$$

defined by $v \mapsto \sum_{\gamma\in\Gamma} (\gamma \cdot v) \otimes \gamma$. The left isomorphism in (10) is constructed using this $\mathbb{C}$-linear identification. Combining (9) and (10) we have

$$H^i(Y, \widehat{E} \otimes \widehat{F}) = 0 \tag{11}$$

for all i. From this it can be deduced that the vector bundle $\widehat{E}$ is semistable. Indeed, using Riemann–Roch and (11) it follows that $\mu(\widehat{E} \otimes \widehat{F}) = \text{genus}(Y) - 1$ (here $\mu(W') = \text{degree}(W')/\text{rank}(W')$ for a vector bundle W'). Therefore, using Riemann–Roch, for any subbundle $W \subset \widehat{E}$, with $\mu(W) > \mu(\widehat{E})$, we have $\chi(W \otimes \widehat{F}) > 0$. Hence for such a subbundle we have $0 < \dim H^0(Y, W \otimes \widehat{F}) \leqslant \dim H^0(Y, \widehat{E} \otimes \widehat{F})$, which contradicts (11). Hence $\widehat{E}$ is a semistable vector bundle.

Since $\widehat{E}$ is semistable, from [2, p. 318, Lemma 3.13] it follows that the parabolic vector bundle E_* is parabolic semistable.

To prove the converse, assume that E_* is parabolic semistable. Therefore, the corresponding Γ-linearized vector bundle $\widehat{E}$ on Y is semistable; see [2, p. 318, Lemma 3.13] and [2, p. 308, Lemma 2.7]. Since $\widehat{E}$ is a semistable vector bundle, a criterion due to Faltings says that there is a vector bundle F on Y such that

$$H^i(Y, \widehat{E} \otimes F) = 0 \tag{12}$$

for all i; see [4, p. 514, Theorem 1.2] and [4, p. 516, Remark]. Set

$$\widetilde{F} := \bigoplus_{\gamma \in \Gamma} \gamma^* F.$$

Using the Γ-linearization of $\widehat{E}$ we have $\gamma^*\widehat{E} = \widehat{E}$ for all $\gamma \in \Gamma$. Hence from (12) it follows that $\dim H^i(Y, \widehat{E} \otimes \widetilde{F}) = (\#\Gamma) \cdot \dim H^i(Y, \widehat{E} \otimes F) = 0$ for all i. Therefore,

$$H^i(Y, \widehat{E} \otimes \widetilde{F} \otimes \widehat{V}) = H^i(Y, \widehat{E} \otimes \widetilde{F}) \otimes_{\mathbb{C}} \mathbb{C}(\Gamma) = 0 \tag{13}$$

for all i, where $\widehat{V}$ is constructed in (6).

The vector bundle $\widetilde{F}$ has a natural Γ-linearization. Let F'_* be the corresponding parabolic vector bundle on X. Let F_* be the parabolic vector bundle obtained from F'_* by forgetting its parabolic structure over $X \setminus S$ and keeping the underlying vector bundle unchanged. Since E_* and V_* do not have any parabolic points outside S, the vector bundle underlying the parabolic tensor product $E_* \otimes F'_* \otimes V_*$ is identified with that of $E_* \otimes F_* \otimes V_*$. Now from (8) and (13) we have $H^i(X, (E_* \otimes F_* \otimes V_*)_0) = 0$ for all i, where $(E_* \otimes F_* \otimes V_*)_0$ is the vector bundle underlying the parabolic tensor product $E_* \otimes F_* \otimes V_*$. This completes the proof of the theorem. □

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