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Harmonic Analysis/Mathematical Analysis

Interpolation by functions with small spectra

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Abstract

We show that if Λ is a 'generic' separated sequence of reals, then there is an unbounded set S of arbitrary small measure (union of some neighborhoods of integers) such that every function on Λ with certain decay condition, can be interpolated by an L^2-function with the spectrum on S (Theorem 1). This should be contrasted against results for compact spectra (Theorems 2 and 3). ***To cite this article: A. Olevskii, A. Ulanovskii, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Interpolation par des fonctions à petits spectres. Nous montrons que si Λ est une suite réelle « générique », il existe un ensemble S de mesure arbitrairement petite et non borné (réunion de voisinages d'entiers) tel que toute fonction à décroissance convenable sur Λ soit prolongeable sur $\mathbb{R}$ en une fonction de carré integrable dont le spectre est dans S (Théorème 1). Cela doit être comparé aux résultats concernant les spectres compacts (Théorèmes 2 et 3). ***Pour citer cet article : A. Olevskii, A. Ulanovskii, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Results

Let $\Lambda = \{\cdots < \lambda_{j-1} < \lambda_j < \lambda_{j+1} < \cdots,\ j \in \mathbb{Z}\}$ be a real sequence. We shall assume that it is separated, i.e. $\inf_j(\lambda_j - \lambda_{j-1}) > 0$. By $D^+(\Lambda)$ we denote the upper uniform density of Λ (see [2, p. 303], [1,3]):

$$D^+(\Lambda) := \lim_{l\to\infty} \max_{a\in\mathbb{R}} \frac{\#(\Lambda \cap (a, a+l))}{l}.$$

Given a space of complex sequences $X = \{c_j,\ j \in \mathbb{Z}\}$, we shall say that a set $S \subset \mathbb{R}$ is an interpolation spectrum for X, if for every $\{c_j\} \in X$ there is a function $F \in L^2(S)$ whose Fourier transform $\hat{F}$ satisfies:

$$\hat{F}(\lambda_j) = c_j, \quad j \in \mathbb{Z}. \tag{1}$$

The case $X = l^2$ is classical. Kahane [2] proved that for a *single interval* S to be interpolation spectrum, it is necessary that $\operatorname{mes} S \geqslant 2\pi D^+(\Lambda)$, and it is sufficient that $\operatorname{mes} S > 2\pi D^+(\Lambda)$. We mention also Beurling's result [1],

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who proved that the last condition is necessary and sufficient for interpolation of l^∞ by functions bounded on $\mathbb{R}$ with spectra on an interval S.

Simple examples show that the sufficient condition above fails already when S is a union of several intervals. However, using a new approach, Landau [3] proved that the necessary condition in Kahane's result still holds for every *bounded* set S.

In the present note we show that if S is *unbounded* and X is a space of 'slowly decreasing sequences', then no such necessary condition may exist. For 'generic' Λ we construct interpolation spectra of arbitrary small measure:

Theorem 1. *Let a separated sequence Λ be linearly independent (mod π) over the field of rational numbers. Then for every $\delta > 0$ there is a set S, a union of some of intervals centered at integers, such that*:

(i) $\operatorname{mes} S < \delta$;
(ii) *for every sequence $c_j = \mathrm{O}(|j|^{-\alpha})$, $\alpha > 1$, there is a function $F \in L^2(S)$ satisfying* (1).

However, if S is a compact set, an analogue of classical results holds even for spaces X of sequences having a 'very fast decay'.

In the next result we suppose that the sequence Λ is distributed 'regularly', i.e. the limit

$$D(\Lambda) := \lim_{l\to\infty} \frac{\#(\Lambda \cap (a, a+l))}{l}$$

exists uniformly with respect to a.

Theorem 2. *Let S be a compact set. If for every sequence $c_j = \mathrm{O}(\mathrm{e}^{-|j|^\alpha})$, $0 < \alpha < 1$, there exists $F \in L^2(S)$ satisfying* (1)*, then* $\operatorname{mes} S \geqslant 2\pi D(\Lambda)$.

We also prove a version of Landau's result for 'interpolation with error'. Denote by $\{\mathbf{e}_j,\ j \in \mathbb{Z}\}$ the standard orthonormal basis in l^2.

Theorem 3. *Let S be a compact set, Λ be a separated sequence and $0 < d < 1$. Suppose there is a sequence of functions $F_j \in L^2(S)$, $\sup_j \|F_j\| < \infty$, such that $\|\hat{F}_j|_\Lambda - \mathbf{e}_j\|_{l^2(\mathbb{Z})} \leqslant d$ for all $j \in \mathbb{Z}$. Then*

$$\operatorname{mes} S \geqslant 2\pi\left(1 - d^2\right) D^+(\Lambda). \tag{2}$$

The bound (2) is sharp for every d.

2. Proof of Theorem 1

Here we shall sketch the proof of Theorem 1. It consists of several steps.

1. Without loss of generality we may assume that $\alpha < 2$. Fix any number β, $1 < \beta < \alpha$. Set

$$S := \bigcup_{j\in\mathbb{Z}} S_j, \quad S_j := (-M_j - 5\gamma_j, -M_j + 5\gamma_j) \cup (M_j - 5\gamma_j, M_j + 5\gamma_j), \tag{3}$$

where

$$\gamma_j := \frac{\gamma}{1 + |j|^\beta},$$

the sequence M_j will be specified in step 4, and γ is any small positive number such that $\operatorname{mes} S < \delta$.

2. Set

$$\Lambda_k := (\Lambda - \lambda_k) \setminus \{0\}, \quad k \in \mathbb{Z}.$$

The independence condition on Λ implies, by Kronecker's theorem, that for every $N > 0$ the subgroup $\{m\lambda\ (\operatorname{mod} \pi)$, $\lambda \in \Lambda_k \cap [-N, N], m \in \mathbb{Z}\}$ is dense in the l−dimensional torus, l being the number of elements in $\Lambda_k \cap [-N, N]$. Hence, the l numbers $|\cos(Mx)|$, $x \in \Lambda_k \cap [-N, N]$, can be made as small as we like by choosing appropriate $M \in \mathbb{N}$.

3. Set

$$g_j(x) := \cos\bigl(M_j(x-\lambda_j)\bigr)\left(\frac{\sin\gamma_j(x-\lambda_j)}{\gamma_j(x-\lambda_j)}\right)^5.$$

The spectrum of g_j belongs to S_j, and we have

$$g_j(\lambda_j) = 1, \tag{4}$$

and

$$\|g_j\|^2_{L^2(\mathbb{R})} \leqslant \mathrm{const}\cdot\bigl(1+|j|^\beta\bigr), \quad j \in \mathbb{Z}. \tag{5}$$

4. By Step 2, the first factor in the definition of g_j can be made arbitrarily small for $\lambda \neq \lambda_j, |\lambda - \lambda_j| < N_j$. By using N_j large enough, one may check that for every positive $\epsilon > 0$ there exists a sequence $M_j \in \mathbb{N}$ such the functions g_j are small on $\Lambda \setminus \{\lambda_j\}$ in the sense that

$$\bigl|g_j(\lambda_k)\bigr| \leqslant \frac{\epsilon}{(1+j^2)(1+(j-k)^4)}, \quad k \neq j, k, j \in \mathbb{Z}. \tag{6}$$

5. Given a sequence $\mathbf{c} = \{c_j,\ j \in \mathbb{Z}\}$, set

$$\|\mathbf{c}\|^2_\beta := \sum_{j=-\infty}^{\infty} |c_j|^2\bigl(1+|j|^\beta\bigr).$$

Let l^2_β denote the weighted space of all sequences $\mathbf{c}$, $\|\mathbf{c}\|_\beta < \infty$. Using (6) and (4), one may check that the linear operator R defined by

$$R\mathbf{e}_j := \sum_{k=-\infty}^{\infty} g_j(\lambda_k)\mathbf{e}_k - \mathbf{e}_j, \quad j \in \mathbb{Z},$$

is well defined on l^2_β. Moreover, if ϵ in (6) is small enough, the norm of this operator in l^2_β is less than 1. It follows that the operator $T := I + R$ is invertible in l^2_γ, where I is the identity operator. We conclude that for every $\mathbf{c} \in l^2_\beta$ the interpolation problem (1) has a solution F whose Fourier transform is given by

$$\hat{F}(x) = \sum_{j\in\mathbb{Z}} b_j g_j(x), \quad \{b_j\} = T^{-1}\mathbf{c} \in l^2_\beta.$$

Also, by (3) and (5), we see that $F \in L^2(S)$.

Remarks.

1. Let ξ_j, $j \in \mathbb{Z}$, be independent identically distributed random variables having a continuous distribution function concentrated on some neighborhood of the origin. By Theorem 1, the random sequence $\Lambda = \{n + \xi_n, n \in \mathbb{Z}\}$ has the property that for each $\delta > 0$, with probability one there exists a random set S, mes $S < \delta$, such that each sequence $c_j = \mathrm{O}(|j|^{-\alpha})$, $\alpha > 1$, can be interpolated by an L^2-function f with the spectrum in S.
2. The decay assumption in Theorem 1 cannot be replaced by $\mathbf{c} \in l^2$. Let Λ be the random sequence above and $X = l^2$. Then one can show that with probability one no set S, mes $S < 2\pi$, can serve as an interpolation spectrum for X.

3. Compact spectra: interpolation with error

Here we sketch a proof of Theorem 3.

1. Claim: Let $0 < c < 1$ and W be a linear subspace of the Paley–Wiener space $PW(-\pi,\pi)$, which is 'c-concentrated on some set Q' in the sense that

$$\int_Q |f|^2 > c\|f\|^2_{L^2(\mathbb{R})}, \quad f \in W.$$

Then

$$\dim W \leqslant \frac{1}{c} \operatorname{mes} Q + 1.$$

This follows from Landau's Lemma 1 (compare (iii) and (viii) in [3], p. 41).

2. Fix a small number $b > 0$ and set $S_b := S + (-b, b)$. Let Φ be any infinitely smooth function supported on $(-b, b)$ satisfying $\hat{\Phi}(0) = 1$ and $|\hat{\Phi}(x)| < 1, x \neq 0$. Set

$$G_j(t) := F_j(t) * \left(\mathrm{e}^{-\mathrm{i}\lambda_j t} \Phi(t)\right).$$

Set $f_j = \hat{F}_j$ and $g_j = \hat{G}_j$. Clearly, each $g_j|_\Lambda$ approximates $\mathbf{e}_j$ with an l^2-error $\leqslant d$. One can prove that if N is sufficiently large, then the space Z spanned by g_j when $|\lambda_j| < N$, is c'-concentrated on the interval $J := (1 + b)(-N, N)$, where c' can be chosen arbitrary close to 1. Hence, for all large N, the space Y of the inverse Fourier transform of the functions $g \cdot 1_J$, $g \in Z$, is c-concentrated on S_b, again with c arbitrary close to 1. The claim above, after re-scaling, gives:

$$\dim Y \leqslant \frac{(1+b)N}{\pi c} \operatorname{mes} S_b + 1.$$

3. Fix a large number N, and denote by $\nu = \nu(N)$ the number of points of Λ in $(-N, N)$. Define vectors $\mathbf{v}_j$ in the Euclidean space $\mathbb{C}^\nu$ by

$$\mathbf{v}_j(l) := g_j(\lambda_l), \quad |\lambda_l| < N.$$

Let V be the linear span of $\mathbf{v}_j$ in $\mathbb{C}^\nu$. Clearly, $\dim Y \geqslant \dim V$. On the other hand, each of $\mathbf{v}_j$ approximates the corresponding $\mathbf{e}_j$ with an error $\leqslant d$. A well-known estimate of the Kolmogorov width of octahedron implies

$$\dim V \geqslant \left(1 - d^2\right)\nu.$$

4. Combining the last three inequalities, one obtains an estimate of ν. The previous argument can be repeated for each interval $(a - N, a + N)$, uniformly over a. Hence, taking the limit as $N \to \infty$, we get an estimate of $D^+(\Lambda)$. Finally, taking the limit as $b \to 0$ and $c \to 1$, we obtain (2).

Theorem 2 can be proved basically by the same argument (for regularly distributed Λ). Observe that the decay restriction in Theorem 2 can be replaced by any non quasi-analytic one.

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