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Differential Geometry

New compatibility conditions for the fundamental theorem of surface theory

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Abstract

The fundamental theorem of surface theory classically asserts that, if a field of positive-definite symmetric matrices $(a_{\alpha\beta})$ of order two and a field of symmetric matrices $(b_{\alpha\beta})$ of order two together satisfy the Gauss and Codazzi–Mainardi equations in a connected and simply-connected open subset ω of $\mathbb{R}^2$, then there exists an immersion $\boldsymbol{\theta}:\omega\to\mathbb{R}^3$ such that these fields are the first and second fundamental forms of the surface $\boldsymbol{\theta}(\omega)$ and this surface is unique up to proper isometries in $\mathbb{R}^3$.

In this Note, we identify new compatibility conditions, expressed again in terms of the functions $a_{\alpha\beta}$ and $b_{\alpha\beta}$, that likewise lead to a similar existence and uniqueness theorem. These conditions take the form

$$\partial_1\mathbf{A}_2-\partial_2\mathbf{A}_1+\mathbf{A}_1\mathbf{A}_2-\mathbf{A}_2\mathbf{A}_1=\mathbf{0}\quad\text{in }\omega,$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are antisymmetric matrix fields of order three that are functions of the fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$, the field $(a_{\alpha\beta})$ appearing in particular through its square root. The unknown immersion $\boldsymbol{\theta}:\omega\to\mathbb{R}^3$ is found in the present approach in function spaces 'with little regularity', viz., $W^{2,p}_{\text{loc}}(\omega;\mathbb{R}^3)$, $p>2$. ***To cite this article: P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

De nouvelles conditions de compatibilité pour le théorème fondamental de la théorie des surfaces. Le théorème fondamental de la théorie des surfaces affirme classiquement que, si un champ de matrices $(a_{\alpha\beta})$ symétriques définies positives d'ordre deux et un champ de matrices $(b_{\alpha\beta})$ symétriques d'ordre deux satisfont ensemble les équations de Gauss et Codazzi–Mainardi dans un ouvert $\omega\subset\mathbb{R}^2$ connexe et simplement connexe, alors il existe une immersion $\boldsymbol{\theta}:\omega\to\mathbb{R}^3$ telle que ces deux champs soient les première et deuxième formes fondamentales de la surface $\boldsymbol{\theta}(\omega)$, et cette surface est unique aux isométries propres de $\mathbb{R}^3$ près.

Dans cette Note, nous identifions de nouvelles conditions de compatibilité, exprimées à nouveau à l'aide des fonctions $a_{\alpha\beta}$ et $b_{\alpha\beta}$, qui conduisent aussi à un théorème analogue d'existence et d'unicité. Ces conditions sont de la forme

$$\partial_1\mathbf{A}_2-\partial_2\mathbf{A}_1+\mathbf{A}_1\mathbf{A}_2-\mathbf{A}_2\mathbf{A}_1=\mathbf{0}\quad\text{dans }\omega,$$

où $\mathbf{A}_1$ et $\mathbf{A}_2$ sont des champs de matrices antisymétriques d'ordre trois, qui sont des fonctions des champs $(a_{\alpha\beta})$ et $(b_{\alpha\beta})$, le champ $(a_{\alpha\beta})$ apparaissant en particulier par l'intermédiaire de sa racine carrée. L'immersion inconnue $\boldsymbol{\theta}:\omega\to\mathbb{R}^3$ est trouvée dans cette

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approche dans des espaces fonctionnelles « de faible régularité », à savoir $W^{2,p}_{\mathrm{loc}}(\omega;\mathbb{R}^3)$, $p>2$. ***Pour citer cet article : P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. The fundamental theorem of surface theory: Classical formulation and recent extensions

To begin with, we list various conventions, notations, and definitions. Greek indices and exponents range in the set $\{1,2\}$ and the summation convention with respect to repeated indices or exponents is used in conjunction with this rule.

All matrices considered in this paper are real. The notations $\mathbb{M}^n$, $\mathbb{M}^n_+$, $\mathbb{S}^n$, $\mathbb{S}^n_>$, $\mathbb{A}^n$, and $\mathbb{O}^n_+$ respectively designate the sets of all square matrices of order n, of all matrices $\mathbf{F}\in\mathbb{M}^n$ with $\det\mathbf{F}>0$, of all symmetric matrices, of all positive-definite symmetric matrices, of all antisymmetric matrices, and of all proper orthogonal matrices, of order n. Given a matrix $\mathbf{A}\in\mathbb{M}^n$, $[\mathbf{A}]_j$ denotes its j-th column vector. The Euclidean norm of $\mathbf{a}\in\mathbb{R}^n$, the Euclidean inner-product of $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$, and the vector product of $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$ are respectively denoted $|\mathbf{a}|$, $\mathbf{a}\cdot\mathbf{b}$, and $\mathbf{a}\wedge\mathbf{b}$.

Given any matrix $\mathbf{C}\in\mathbb{S}^n_>$, there exists one and only one matrix $\mathbf{U}\in\mathbb{S}^n_>$ such that $\mathbf{U}^2=\mathbf{C}$. The matrix $\mathbf{U}$, which is then denoted $\mathbf{C}^{1/2}$, is the *square root* of $\mathbf{C}$. Any invertible matrix $\mathbf{F}\in\mathbb{M}^n_+$ admits a unique *polar factorization* $\mathbf{F}=\mathbf{RU}$, as a product of a matrix $\mathbf{R}\in\mathbb{O}^n_+$ by a matrix $\mathbf{U}\in\mathbb{S}^n_>$, with $\mathbf{U}=(\mathbf{F}^{\mathrm{T}}\mathbf{F})^{1/2}$ and $\mathbf{R}=\mathbf{FU}^{-1}$.

The coordinates of a point $y\in\mathbb{R}^2$ are denoted y_α and partial derivatives of the first and second order, in the usual sense or in the sense of distributions, are denoted $\partial_\alpha:=\partial/\partial y_\alpha$ and $\partial_{\alpha\beta}:=\partial^2/\partial y_\alpha\partial y_\beta$.

Let ω be an open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta}\in\mathcal{C}^3(\omega;\mathbb{R}^3)$ be an immersion. Let

$$a_{\alpha\beta}:=\partial_\alpha\boldsymbol{\theta}\cdot\partial_\beta\boldsymbol{\theta}\in\mathcal{C}^2(\omega)\quad\text{and}\quad b_{\alpha\beta}:=\partial_{\alpha\beta}\boldsymbol{\theta}\cdot\frac{\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}|}\in\mathcal{C}^1(\omega)$$

denote the components of the first and second fundamental forms of the surface $\boldsymbol{\theta}(\omega)$, and let

$$\Gamma_{\alpha\beta\tau}:=\frac{1}{2}(\partial_\beta a_{\alpha\tau}+\partial_\alpha a_{\beta\tau}-\partial_\tau a_{\alpha\beta}),\quad \big(a^{\sigma\tau}\big):=(a_{\alpha\beta})^{-1},\quad\text{and}\quad \Gamma^\sigma_{\alpha\beta}:=a^{\sigma\tau}\Gamma_{\alpha\beta\tau}.$$

Then it is well known that the functions $a_{\alpha\beta}$ and $b_{\alpha\beta}$ necessarily satisfy *compatibility conditions*, which take the form of the *Gauss and Codazzi–Mainardi equations*, viz.,

$$\partial_\beta\Gamma_{\alpha\sigma\tau}-\partial_\sigma\Gamma_{\alpha\beta\tau}+\Gamma^\mu_{\alpha\beta}\Gamma_{\sigma\tau\mu}-\Gamma^\mu_{\alpha\sigma}\Gamma_{\beta\tau\mu}=b_{\alpha\sigma}b_{\beta\tau}-b_{\alpha\beta}b_{\sigma\tau}\quad\text{in }\omega,$$
$$\partial_\beta b_{\alpha\sigma}-\partial_\sigma b_{\alpha\beta}+\Gamma^\mu_{\alpha\sigma}b_{\beta\mu}-\Gamma^\mu_{\alpha\beta}b_{\sigma\mu}=0\quad\text{in }\omega.$$

The Gauss equations reduce to only one equation (corresponding, e.g., to $\alpha=1$, $\beta=2$, $\sigma=1$, $\tau=2$) and the Codazzi–Mainardi equations reduce to only two equations (corresponding, e.g., to $\alpha=1$, $\beta=2$, $\sigma=1$ and $\alpha=1$, $\beta=2$, $\sigma=2$).

It is also well known that, if a field of positive-definite symmetric matrices $(a_{\alpha\beta})\in\mathcal{C}^2(\omega;\mathbb{S}^2_>)$ and a field of symmetric matrices $(b_{\alpha\beta})\in\mathcal{C}^1(\omega;\mathbb{S}^2)$ verify the Gauss and Codazzi–Mainardi equations and if the set ω is *simply-connected*, then conversely, *there exists an immersion $\boldsymbol{\theta}\in\mathcal{C}^3(\omega;\mathbb{R}^3)$ such that $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ are the first and second fundamental forms of the surface $\boldsymbol{\theta}(\omega)$*. If the set ω is in addition *connected*, then such an immersion is *uniquely defined up to proper isometries of $\mathbb{R}^3$*.

These existence and uniqueness results constitute together the *fundamental theorem of surface theory*, which goes back to Janet [8] and Cartan [1] (for a self-contained, and essentially elementary, proof, see [5] or [2, Chapter 2]). Its regularity assumptions have since then been significantly weakened: First, Hartman & Wintner [7] have shown that this theorem still holds if the fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ are only of class $\mathcal{C}^1$ and $\mathcal{C}^0$, with a resulting immersion in the space $\mathcal{C}^2(\omega;\mathbb{R}^3)$. Then S. Mardare further relaxed these assumptions, first in [9] to fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ of class $W^{1,\infty}_{\mathrm{loc}}$ and L^∞_{loc}, then in [10] to fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ of class $W^{1,p}_{\mathrm{loc}}$ and L^p_{loc} for some $p>2$, with resulting immersions in the spaces $W^{2,\infty}_{\mathrm{loc}}(\omega;\mathbb{R}^3)$ and $W^{2,p}_{\mathrm{loc}}(\omega;\mathbb{R}^3)$, respectively. Naturally, the Gauss and Codazzi–Mainardi relations are only satisfied in the sense of distributions in such cases.

2. New compatibility conditions satisfied by the two fundamental forms of a surface

Our objective is to identify *new compatibility conditions* satisfied by the first and second fundamental forms of a surface $\boldsymbol{\theta}(\omega)$ that share the same properties: They are necessary (Theorem 2.1) and they are sufficient for the existence of the immersion $\boldsymbol{\theta}:\omega\to\mathbb{R}^3$ if ω is simply-connected (Theorem 3.1).

Theorem 2.1. *Let ω be an open subset of $\mathbb{R}^2$, let $p>2$, and let $\boldsymbol{\theta}\in W^{2,p}_{\mathrm{loc}}(\omega;\mathbb{R}^3)$ be an immersion. Define the vector fields $\mathbf{a}_i\in W^{1,p}_{\mathrm{loc}}(\omega;\mathbb{R}^3)$, $1\leqslant i\leqslant 3$, and the matrix fields $(a_{\alpha\beta})\in W^{1,p}_{\mathrm{loc}}(\omega;\mathbb{S}^2_>)$ and $(b_{\alpha\beta})\in L^p_{\mathrm{loc}}(\omega;\mathbb{S}^2)$ by*

$$\mathbf{a}_\alpha:=\partial_\alpha\boldsymbol{\theta}\quad\text{and}\quad \mathbf{a}_3:=\frac{\mathbf{a}_1\wedge\mathbf{a}_2}{|\mathbf{a}_1\wedge\mathbf{a}_2|},\qquad a_{\alpha\beta}:=\mathbf{a}_\alpha\cdot\mathbf{a}_\beta\quad\text{and}\quad b_{\alpha\beta}:=\partial_\alpha\mathbf{a}_\beta\cdot\mathbf{a}_3.$$

Define also the matrix fields $(a^{\sigma\tau})\in W^{1,p}_{\mathrm{loc}}(\omega;\mathbb{S}^2_>)$ and the functions $\Gamma_{\alpha\beta\tau}\in L^p_{\mathrm{loc}}(\omega)$, $\Gamma^\sigma_{\alpha\beta}\in L^p_{\mathrm{loc}}(\omega)$, and $b^\sigma_\alpha\in L^p_{\mathrm{loc}}(\omega)$ by

$$\left(a^{\sigma\tau}\right):=(a_{\alpha\beta})^{-1},\quad \Gamma_{\alpha\beta\tau}:=\frac{1}{2}(\partial_\beta a_{\alpha\tau}+\partial_\alpha a_{\beta\tau}-\partial_\tau a_{\alpha\beta}),\quad \Gamma^\sigma_{\alpha\beta}:=a^{\sigma\tau}\Gamma_{\alpha\beta\tau},\quad b^\sigma_\alpha:=a^{\beta\sigma}b_{\alpha\beta}.$$

Finally, define the matrix fields $\boldsymbol{\Gamma}_\alpha\in L^p_{\mathrm{loc}}(\omega;\mathbb{M}^3)$, $\mathbf{C}\in W^{1,p}_{\mathrm{loc}}(\omega;\mathbb{S}^3_>)$, $\mathbf{U}\in W^{1,p}_{\mathrm{loc}}(\omega;\mathbb{S}^3_>)$, and $\mathbf{A}_\alpha\in L^p_{\mathrm{loc}}(\omega;\mathbb{M}^3)$ by

$$\boldsymbol{\Gamma}_\alpha:=\begin{pmatrix}\Gamma^1_{\alpha1} & \Gamma^1_{\alpha2} & -b^1_\alpha\\ \Gamma^2_{\alpha1} & \Gamma^2_{\alpha2} & -b^2_\alpha\\ b_{\alpha1} & b_{\alpha2} & 0\end{pmatrix},\qquad \mathbf{C}:=\begin{pmatrix}a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & 1\end{pmatrix},$$
$$\mathbf{U}:=\mathbf{C}^{1/2},\qquad \mathbf{A}_\alpha:=(\mathbf{U}\boldsymbol{\Gamma}_\alpha-\partial_\alpha\mathbf{U})\mathbf{U}^{-1}.$$

Then the matrix fields $\mathbf{A}_\alpha$ are antisymmetric and they necessarily satisfy the following compatibility conditions:

$$\partial_1\mathbf{A}_2-\partial_2\mathbf{A}_1+\mathbf{A}_1\mathbf{A}_2-\mathbf{A}_2\mathbf{A}_1=\mathbf{0}\quad\text{in } \mathcal{D}'\left(\omega;\mathbb{A}^3\right).$$

Sketch of proof. (i) *Technical preliminaries.* These preliminaries consist in showing that the following regularities and equations hold:

$$\mathbf{a}_3\in W^{1,p}_{\mathrm{loc}}\left(\omega;\mathbb{R}^3\right)\quad\text{and}\quad \left(a^{\sigma\tau}\right)\in W^{1,p}_{\mathrm{loc}}\left(\omega;\mathbb{S}^2_>\right),\quad \mathbf{U}\in W^{1,p}_{\mathrm{loc}}\left(\omega;\mathbb{S}^3_>\right)\quad\text{and}\quad \mathbf{U}^{-1}\in W^{1,p}_{\mathrm{loc}}\left(\omega;\mathbb{S}^3_>\right),$$
$$\partial_\alpha\mathbf{a}_\beta=\Gamma^\sigma_{\alpha\beta}\mathbf{a}_\sigma+b_{\alpha\beta}\mathbf{a}_3\quad\text{and}\quad \partial_\alpha\mathbf{a}_3=-b^\sigma_\alpha\mathbf{a}_\sigma\quad\text{in } L^p_{\mathrm{loc}}\left(\omega;\mathbb{R}^3\right)$$

(of course, the last relations are nothing but the extensions of the classical *formulas of Gauss* and *Weingarten* to function spaces with little regularity).

(ii) *Introduction of the antisymmetric matrix fields* $\mathbf{A}_\alpha$. One proves that the matrix field $\mathbf{F}\in W^{1,p}_{\mathrm{loc}}(\omega;\mathbb{M}^3_+)$ defined by $[\mathbf{F}]_j:=\mathbf{a}_j$, $1\leqslant j\leqslant 3$, satisfies

$$\mathbf{C}=\mathbf{F}^{\mathrm{T}}\mathbf{F}\text{ in } W^{1,p}_{\mathrm{loc}}\left(\omega;\mathbb{S}^3_>\right),\quad \partial_\alpha\mathbf{F}=\mathbf{F}\boldsymbol{\Gamma}_\alpha\text{ in } L^p_{\mathrm{loc}}\left(\omega;\mathbb{M}^3\right),\quad\text{and}\quad \mathbf{F}(y)\in\mathbb{M}^3_+\text{ for all } y\in\omega$$

(the second relation is simply a re-writing in matrix form of the formulas of Gauss and Weingarten; this observation is due to S. Mardare [9]).

At each point $y\in\omega$, let $\mathbf{F}(y)=\mathbf{R}(y)\mathbf{U}(y)$ denote the polar factorization of the matrix $\mathbf{F}(y)\in\mathbb{M}^3_+$. Then $\mathbf{R}\in W^{1,p}_{\mathrm{loc}}(\omega;\mathbb{O}^3_+)$ and

$$\partial_\alpha\mathbf{R}=\mathbf{R}\mathbf{A}_\alpha\quad\text{in } L^p_{\mathrm{loc}}\left(\omega;\mathbb{M}^3\right)\text{ where } \mathbf{A}_\alpha:=(\mathbf{U}\boldsymbol{\Gamma}_\alpha-\partial_\alpha\mathbf{U})\mathbf{U}^{-1}\in L^p_{\mathrm{loc}}\left(\omega;\mathbb{M}^3\right).$$

The relations $\mathbf{I}=\mathbf{R}^{\mathrm{T}}\mathbf{R}$ and $\partial_\alpha\mathbf{R}=\mathbf{R}\mathbf{A}_\alpha$ then imply that $\mathbf{0}=(\partial_\alpha\mathbf{R})^{\mathrm{T}}\mathbf{R}+\mathbf{R}^{\mathrm{T}}\partial_\alpha\mathbf{R}=\mathbf{A}^{\mathrm{T}}_\alpha+\mathbf{A}_\alpha$ in ω. Therefore the matrix fields $\mathbf{A}_\alpha$ are *antisymmetric*.

(iii) *Compatibility conditions satisfied by the matrix fields* $\mathbf{A}_\alpha$. The relations $\partial_\alpha\mathbf{R}=\mathbf{R}\mathbf{A}_\alpha$ in $L^p_{\mathrm{loc}}(\omega;\mathbb{M}^3)$ satisfied by the matrix fields $\mathbf{R}\in W^{1,p}_{\mathrm{loc}}(\omega;\mathbb{O}^3_+)$ and $\mathbf{A}_\alpha\in L^p_{\mathrm{loc}}(\omega;\mathbb{A}^3)$ found in part (ii) imply that

$$\partial_{\beta\alpha}\mathbf{R}=(\partial_\beta\mathbf{R})\mathbf{A}_\alpha+\mathbf{R}\partial_\beta\mathbf{A}_\alpha=\mathbf{R}\mathbf{A}_\beta\mathbf{A}_\alpha+\mathbf{R}\partial_\beta\mathbf{A}_\alpha\quad\text{in } \mathcal{D}'\left(\omega;\mathbb{M}^3\right),$$
$$\partial_{\alpha\beta}\mathbf{R}=(\partial_\alpha\mathbf{R})\mathbf{A}_\beta+\mathbf{R}\partial_\alpha\mathbf{A}_\beta=\mathbf{R}\mathbf{A}_\alpha\mathbf{A}_\beta+\mathbf{R}\partial_\alpha\mathbf{A}_\beta\quad\text{in } \mathcal{D}'\left(\omega;\mathbb{M}^3\right)$$

(the products $\mathbf{R}\mathbf{A}_\beta\mathbf{A}_\alpha$ are well-defined distributions, since $\mathbf{A}_\beta\mathbf{A}_\alpha \in L^{p/2}_{\mathrm{loc}}(\omega;\mathbb{M}^3)$). Hence the relations $\partial_{\beta\alpha}\mathbf{R} = \partial_{\alpha\beta}\mathbf{R}$ imply that

$$\mathbf{R}\partial_\alpha\mathbf{A}_\beta - \mathbf{R}\partial_\beta\mathbf{A}_\alpha + \mathbf{R}\mathbf{A}_\alpha\mathbf{A}_\beta - \mathbf{R}\mathbf{A}_\beta\mathbf{A}_\alpha = \mathbf{0} \quad \text{in } \mathcal{D}'(\omega;\mathbb{M}^3).$$

Thanks to a series of relations valid in the sense of distributions, it is then shown that the invertibility of the matrices $\mathbf{R}(y)$ at all $y \in \omega$ in turn implies that

$$\partial_\alpha\mathbf{A}_\beta - \partial_\beta\mathbf{A}_\alpha + \mathbf{A}_\alpha\mathbf{A}_\beta - \mathbf{A}_\beta\mathbf{A}_\alpha = \mathbf{0} \quad \text{in } \mathcal{D}'(\omega;\mathbb{A}^3).$$

In order that these relations hold for all $\alpha,\beta \in \{1,2\}$, it clearly suffices that the relation corresponding to $\alpha = 1$ and $\beta = 2$ holds. □

Like the Gauss and Codazzi–Mainardi equations, the compatibility conditions found in Theorem 2.1 as *necessary* conditions reduce to only *three scalar relations*, since the matrix fields $\mathbf{A}_\alpha$ are antisymmetric.

A different set of *necessary* compatibility equations, also related to a rotation field on a surface, has been proposed by Vallée and Fortuné [13].

3. Sufficiency of the compatibility conditions

We next show that, if the open set $\omega \subset \mathbb{R}^2$ is *simply-connected*, the compatibility conditions that were found to be necessary in Theorem 2.1 are also *sufficient*.

Theorem 3.1. *Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$ and let $p > 2$. Let there be given two matrix fields $(a_{\alpha\beta}) \in W^{1,p}_{\mathrm{loc}}(\omega;\mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in L^p_{\mathrm{loc}}(\omega;\mathbb{S}^2)$ that satisfy*

$$\partial_1\mathbf{A}_2 - \partial_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2 - \mathbf{A}_2\mathbf{A}_1 = \mathbf{0} \quad \text{in } \mathcal{D}'(\omega;\mathbb{A}^3),$$

where the matrix fields $\mathbf{A}_\alpha \in L^p_{\mathrm{loc}}(\omega;\mathbb{A}^3)$ are constructed from the matrix fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ by means of the following sequence of definitions:

$$\Gamma_{\alpha\beta\tau} := \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}) \in L^p_{\mathrm{loc}}(\omega), \qquad (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1} \in L^p_{\mathrm{loc}}(\omega;\mathbb{S}^2_>),$$

$$\Gamma^\sigma_{\alpha\beta} := a^{\sigma\tau}\Gamma_{\alpha\beta\tau} \in L^p_{\mathrm{loc}}(\omega), \qquad b^\sigma_\alpha := a^{\beta\sigma}b_{\alpha\beta} \in L^p_{\mathrm{loc}}(\omega),$$

$$\mathbf{\Gamma}_\alpha := \begin{pmatrix} \Gamma^1_{\alpha1} & \Gamma^1_{\alpha2} & -b^1_\alpha \\ \Gamma^2_{\alpha1} & \Gamma^2_{\alpha2} & -b^2_\alpha \\ b_{\alpha1} & b_{\alpha2} & 0 \end{pmatrix} \in L^p_{\mathrm{loc}}(\omega;\mathbb{M}^3), \qquad \mathbf{C} := \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in W^{1,p}_{\mathrm{loc}}(\omega;\mathbb{S}^3_>),$$

$$\mathbf{U} := \mathbf{C}^{1/2} \in W^{1,p}_{\mathrm{loc}}(\omega;\mathbb{S}^3_>), \qquad \mathbf{A}_\alpha := (\mathbf{U}\mathbf{\Gamma}_\alpha - \partial_\alpha\mathbf{U})\mathbf{U}^{-1} \in L^p_{\mathrm{loc}}(\omega;\mathbb{A}^3).$$

Then there exists an immersion $\boldsymbol{\theta} \in W^{2,p}_{\mathrm{loc}}(\omega;\mathbb{R}^3)$ such that

$$a_{\alpha\beta} = \partial_\alpha\boldsymbol{\theta}\cdot\partial_\beta\boldsymbol{\theta} \quad \text{in } W^{1,p}_{\mathrm{loc}}(\omega) \quad \text{and} \quad b_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta}\cdot\frac{\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta}\wedge\partial_2\boldsymbol{\theta}|} \quad \text{in } L^p_{\mathrm{loc}}(\omega).$$

If any other immersion $\tilde{\boldsymbol{\theta}} \in W^{2,p}_{\mathrm{loc}}(\omega;\mathbb{R}^3)$ satisfies the above relations (with $\tilde{\boldsymbol{\theta}}$ in lieu of $\boldsymbol{\theta}$), then there exist a vector $\mathbf{c} \in \mathbb{R}^3$ and a matrix $\mathbf{Q} \in \mathbb{O}^3_+$ such that $\tilde{\boldsymbol{\theta}}(y) = \mathbf{c} + \mathbf{Q}\boldsymbol{\theta}(y)$ for all $y \in \omega$.

Sketch of proof. (i) *The matrix fields $\mathbf{A}_\alpha = (\mathbf{U}\mathbf{\Gamma}_\alpha - \partial_\alpha\mathbf{U})\mathbf{U}^{-1}$ are antisymmetric.* This property is established by a direct computation, based on the definitions of the functions $\Gamma_{\alpha\beta\tau}$, $a^{\sigma\tau}$, $\Gamma^\sigma_{\alpha\beta}$, b^σ_α and of the matrix fields $\mathbf{\Gamma}_\alpha$ and $\mathbf{C}$.

(ii) *Let there be given a point $y_0 \in \omega$ and a matrix $\mathbf{R}_0 \in \mathbb{O}^3_+$. Then there exists one and only one matrix field $\mathbf{R} \in W^{1,p}_{\mathrm{loc}}(\omega;\mathbb{O}^3_+)$ that satisfies $\partial_\alpha\mathbf{R} = \mathbf{R}\mathbf{A}_\alpha$* in $L^p_{\mathrm{loc}}(\omega;\mathbb{M}^3)$ *and* $\mathbf{R}(y_0) = \mathbf{R}_0$. Since the matrix fields $\mathbf{A}_\alpha$ satisfy $\partial_\alpha\mathbf{A}_\beta - \partial_\beta\mathbf{A}_\alpha + \mathbf{A}_\alpha\mathbf{A}_\beta - \mathbf{A}_\beta\mathbf{A}_\alpha = \mathbf{0}$ in $\mathcal{D}'(\omega;\mathbb{A}^3)$, a key existence and uniqueness result of S. Mardare [10, Theorem 7] for linear differential systems with little regularity provides the existence and uniqueness of a solution $\mathbf{R} \in W^{1,p}_{\mathrm{loc}}(\omega;\mathbb{M}^3)$.

In order to show that this matrix field $\mathbf{R}$ is proper orthogonal, we note that the matrix field $\mathbf{R}^{\mathrm{T}}\mathbf{R} \in W^{1,p}_{\mathrm{loc}}(\omega, \mathbb{M}^3)$ satisfies

$$\partial_\alpha\big(\mathbf{R}^{\mathrm{T}}\mathbf{R}\big) = (\partial_\alpha \mathbf{R})^{\mathrm{T}}\mathbf{R} + \mathbf{R}^{\mathrm{T}}\partial_\alpha \mathbf{R} = \mathbf{A}_\alpha^{\mathrm{T}}\big(\mathbf{R}^{\mathrm{T}}\mathbf{R}\big) + \big(\mathbf{R}^{\mathrm{T}}\mathbf{R}\big)\mathbf{A}_\alpha \quad \text{in } L^p_{\mathrm{loc}}\big(\omega; \mathbb{M}^3\big) \quad \text{and} \quad \big(\mathbf{R}^{\mathrm{T}}\mathbf{R}\big)(y_0) = \mathbf{I}.$$

That $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$ is the unique solution to this system is deduced from the uniqueness result of S. Mardare [11, Theorem 4.2]. It then follows that $\det \mathbf{R}(y) = \det \mathbf{R}_0 = 1$ for all $y \in \omega$ since ω is connected.

(iii) *The matrix field* $\mathbf{R} \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{O}^3_+)$ *being that determined in* (ii)*, there exists an immersion* $\boldsymbol{\theta} \in W^{2,p}_{\mathrm{loc}}(\omega, \mathbb{R}^3)$ *that satisfies* $\partial_\alpha \boldsymbol{\theta} = \mathbf{R}\mathbf{u}_\alpha$ *in* $W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{R}^3)$*, where* $\mathbf{u}_\alpha := [\mathbf{U}]_\alpha \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{R}^3)$. Resorting this time to the "Poincaré lemma with little regularity" established in [10, Theorem 8], we conclude that this system has a solution $\boldsymbol{\theta} \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{R}^3)$ because the compatibility relations $\partial_\beta(\mathbf{R}\mathbf{u}_\alpha) = \partial_\alpha(\mathbf{R}\mathbf{u}_\beta)$ in $L^p_{\mathrm{loc}}(\omega; \mathbb{R}^3)$ are satisfied (these relations themselves follow from the relations $\partial_\alpha \mathbf{R} = \mathbf{R}\mathbf{A}_\alpha$ found in part (ii) combined with appropriate computations). Since the fields $\mathbf{R}$ and $\mathbf{u}_\alpha$ are respectively in the spaces $W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{O}^3_+)$ and $W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{R}^3)$, it follows that $\boldsymbol{\theta} \in W^{2,p}_{\mathrm{loc}}(\omega; \mathbb{R}^3)$. Since the vectors $\mathbf{u}_\alpha(y)$ are linearly independent and the matrix $\mathbf{R}(y)$ is proper orthogonal at all points $y \in \omega$, it further follows that $\boldsymbol{\theta}$ is an immersion.

(iv) *The given matrix fields* $(a_{\alpha\beta}) \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{S}^2_>)$ *and* $(b_{\alpha\beta}) \in L^p_{\mathrm{loc}}(\omega; \mathbb{S}^2)$ *are the first and second fundamental forms of the surface* $\boldsymbol{\theta}(\omega)$. Define the matrix and vector fields $\mathbf{F} := \mathbf{R}\mathbf{U} \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{M}^3)$ and $\mathbf{f}_j := [\mathbf{F}]_j \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{R}^3)$, $1 \leqslant j \leqslant 3$, where $\mathbf{R} \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{O}^3_+)$ is the matrix field found in (ii), $\mathbf{U} = \mathbf{C}^{1/2} \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{S}^3_>)$, and the matrix field $\mathbf{C}$ is defined in terms of the functions $a_{\alpha\beta}$ as in the statement of the theorem. Then the conclusions follow from a series of computations, based on the relation $\mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{C}$, the specific form of the matrix field $\mathbf{C}$, and the relations $\mathbf{F} = \mathbf{R}\mathbf{U}$ and $\mathbf{f}_\alpha = \partial_\alpha \boldsymbol{\theta} = \mathbf{R}\mathbf{u}_\alpha$.

(v) The *uniqueness of the immersion* $\boldsymbol{\theta} \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{R}^3)$ *up to proper isometries* follows from the rigidity theorem with little regularity established in [6, Theorem 3]. □

By contrast with the above proof, the proof in the 'classical' approach (once properly extended to spaces with little regularity; cf. S. Mardare [10]) first seeks a matrix field $\mathbf{F} \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{M}^3)$ as a solution of the Pfaff system $\partial_\alpha \mathbf{F} = \mathbf{F}\boldsymbol{\Gamma}_\alpha$, then the sought immersion $\boldsymbol{\theta} \in W^{2,p}_{\mathrm{loc}}(\omega; \mathbb{R}^3)$ as a solution to the system $\partial_\alpha \boldsymbol{\theta} = \mathbf{f}_\alpha$, where $\mathbf{f}_\alpha$ denotes the α-th column vector field of the matrix field $\mathbf{F}$.

An inspection of the proof reveals the *geometric nature of this approach*: Let the *canonical three-dimensional extension* $\boldsymbol{\Theta} : \omega \times \mathbb{R} \to \mathbb{R}^3$ *of an immersion* $\boldsymbol{\theta} \in W^{2,p}_{\mathrm{loc}}(\omega; \mathbb{R}^3)$ be defined by $\boldsymbol{\Theta}(y, x_3) = \boldsymbol{\theta}(y) + x_3\mathbf{a}_3(y)$, for all $y \in \omega$ and $x_3 \in \mathbb{R}$, and let the matrix field $\mathbf{F} \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{M}^3)$ be defined by $\mathbf{F}(y) = \boldsymbol{\nabla}\boldsymbol{\Theta}(y, 0)$. Then the fields $\mathbf{R}$ and $\mathbf{U}$ satisfy $\mathbf{F} = \mathbf{R}\mathbf{U}$ in $W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{M}^3)$. In other words, *the proper orthogonal matrix field* $\mathbf{R}$ *is nothing but the rotation field that appears in the polar factorization of the gradient of the canonical three-dimensional extension* $\boldsymbol{\Theta}$ *of the immersion* $\boldsymbol{\theta}$ *at* $x_3 = 0$.

The above compatibility conditions are in a sense the "surface analogs" of similar "three-dimensional" compatibility conditions satisfied in an open subset Ω of $\mathbb{R}^3$ by the *square root* of the metric tensor field $\boldsymbol{\nabla}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{\nabla}\boldsymbol{\Theta} \in \mathcal{C}^2(\Omega; \mathbb{S}^3_>)$ associated with a given immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$. These three-dimensional conditions, which were first identified (in componentwise form) by Shield [12], have been recently shown to be also *sufficient* for the existence of such an immersion $\boldsymbol{\Theta}$ when the set Ω is simply connected, also in function spaces with little regularity; cf. [4].

As expected, one can also show that the above new compatibility conditions are equivalent to the Gauss and Codazzi–Mainardi equations.

Complete proofs of all the results announced in this Note are found in [3].

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