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Differential Geometry

Canonical semisprays for higher order Lagrange spaces

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Abstract

We show that the canonical semispray of a regular Lagrangian of order k is uniquely determined by two associated Cartan–Poincaré one-forms. Equivalently, the canonical semispray is uniquely determined by its canonical presymplectic structure and one of the Cartan–Poincaré one-forms. We prove that this $k+1$ order vector field is determined by a variational problem, for which only the vertical part of the curve is varied. ***To cite this article: I. Bucataru, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Sémi-gerbes canoniques pour les espaces lagrangiens d'ordre supérieur. Nous obtenons que la sémi-gerbe canonique d'un Lagrangien régulier d'ordre k est uniquement déterminée par deux un-formes Cartan–Poincaré associées. Autrement dit, la sémi-gerbe canonique est uniquement déterminée par sa structure présimplectique canonique et par une des une-formes Cartan–Poincaré. Nous prouvons que ce champ de vecteurs d'ordre $k+1$ est déterminée par un problème variationel pour lequel seulement la partie verticale de la courbe varie. ***Pour citer cet article : I. Bucataru, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

We denote by (T^kM, π^k, M), $k \geqslant 1$, the space of tangent bundle of order k over a smooth, real, n-dimensional manifold M, [2]. Local coordinates (x^i) on M induce local coordinates $(x^i, y^{(1)i}, \ldots, y^{(k)i})$ on T^kM, where for a k-jet $j_0^k\rho \in T^kM$, the coordinate functions are defined as follows

$$y^{(\alpha)i}\left(j_0^k\rho\right) = \frac{1}{\alpha!}\frac{\mathrm{d}^\alpha(x^i \circ \rho)}{\mathrm{d}t^\alpha}\bigg|_{t=0}, \quad \alpha \in \{1, \ldots, k\}.$$

The tangent structures of order k, J and $J^\alpha = \underbrace{J \circ \cdots \circ J}_{\alpha\text{-times}}$, are defined as follows, [3],

$$J^\alpha = \frac{\partial}{\partial y^{(\alpha)i}} \otimes \mathrm{d}x^i + \frac{\partial}{\partial y^{(\alpha+1)i}} \otimes \mathrm{d}y^{(1)i} + \cdots + \frac{\partial}{\partial y^{(\alpha+k-1)i}} \otimes \mathrm{d}y^{(k-1)i}, \quad \forall \alpha \in \{1, 2, \ldots, k\}.$$

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The foliated structure of T^kM allows k regular, integrable, vertical distributions, $V_{k-\alpha+1} = \operatorname{Ker} J^\alpha = \operatorname{Im} J^{k-\alpha+1}$, $\alpha \in \{1, \dots, k\}$. The following k vertical vector fields are globally defined on T^kM, they are called Liouville vector fields and they are defined by induction as follows:

$$\Gamma_k = y^{(1)i}\frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i}\frac{\partial}{\partial y^{(2)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k)i}}, \quad \Gamma_\alpha = J^{k-\alpha}(\Gamma_k), \quad \alpha \in \{1, 2, \dots, k\}.$$

A semispray is a globally defined vector field S on T^kM that satisfies the equation $JS = \Gamma_k$. Therefore, a semispray S, which is a vector field of order $k+1$ can be expressed as follows:

$$S = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}} - (k+1)G^i\big(x, y^{(1)}, \dots, y^{(k)}\big)\frac{\partial}{\partial y^{(k)i}} \tag{1}$$

and it is perfectly determined by its coefficient functions $G^i(x, y^{(1)}, \dots, y^{(k)})$. In this paper we provide conditions that uniquely determine the canonical semispray of a k-th order regular Lagrange space.

Consider $L : T^kM \longrightarrow \mathbb{R}$ a regular Lagrangian of order k, $k \geqslant 1$. In other words, the metric tensor $g_{ij} = (1/2)(\partial^2 L)/(\partial y^{(k)i}\partial y^{(k)j})$ is a symmetric second order tensor field that has maximal rank n on T^kM.

For a regular Lagrangian of order k, we consider the following Cartan–Poincaré one-forms, [4]:

$$\theta^\alpha_L = \mathrm{d}_{J^\alpha} L = \frac{\partial L}{\partial y^{(\alpha)i}}\,\mathrm{d}x^i + \cdots + \frac{\partial L}{\partial y^{(k-\alpha)i}}\,\mathrm{d}y^{(\alpha)i}, \quad \alpha \in \{1, \dots, k\}. \tag{2}$$

We consider also the following Cartan–Poincaré two-forms:

$$\omega^\alpha_L = \mathrm{d}\theta^\alpha_L = \mathrm{d}\left(\frac{\partial L}{\partial y^{(\alpha)i}}\right) \wedge \mathrm{d}x^i + \cdots + \mathrm{d}\left(\frac{\partial L}{\partial y^{(k-\alpha)i}}\right) \wedge \mathrm{d}y^{(\alpha)i}, \quad \alpha \in \{1, \dots, k\}. \tag{3}$$

The Cartan–Poincaré two-form ω^k_L is closed if and only if $k = 1$. Moreover, for $k > 1$, the regularity of the Lagrangian L implies the fact that $\operatorname{rank}(\omega^k_L) = 2n < (k+1)n = \dim(T^kM)$. We refer to ω^k_L as to the canonical presymplectic structure of the Lagrangian L.

Directly from expressions (2) and (3) we obtain the following properties:

Proposition 1.1. *Consider S an arbitrary semispray on T^kM. For a Lagrangian L of order k, its Cartan–Poincaré one and two-forms have the following properties*:

(i) $\mathrm{d}_{J^\alpha}\theta^\beta_L = \mathrm{d}_{J^\alpha}\mathrm{d}_{J^\beta} L = 0$, *for* $\alpha + \beta \geqslant k+1$;
(ii) $i_S\theta^\alpha_L = \Gamma_{k-\alpha+1}(L)$, *for* $\alpha \in \{1, \dots, k\}$;
(iii) $\mathcal{L}_S\theta^\alpha_L = i_S\omega^\alpha_L + \mathrm{d}(\Gamma_{k-\alpha+1}(L))$, *for* $\alpha \in \{1, \dots, k\}$.

The first property of Proposition 1.1 means that $\omega^k_L(J^\alpha X, J^\alpha Y) = 0$, $\forall\alpha \in \{1, \dots, k\}$ and $\forall X, Y \in \chi(T^kM)$ and therefore all vertical distributions V_α are Lagrangian subbundles for the presymplectic structure ω^k_L.

2. Canonical semispray

Consider the following variational problem for the regular Lagrangian L. Let

$$c_\varepsilon : t \in I \mapsto c_\varepsilon(t) = \left(x^i(t), \frac{\mathrm{d}x^i}{\mathrm{d}t}(t), \dots, \frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}x^i}{\mathrm{d}t^{k-1}} + \varepsilon V^i(t), \frac{1}{k!}\frac{\mathrm{d}^k x^i}{\mathrm{d}t^k}(t) + \varepsilon\frac{\mathrm{d}V^i}{\mathrm{d}t}(t)\right) \in T^kM,$$

be a variation of a curve $c(t) = c_0(t)$, where ε belongs to some small neighborhood of $0 \in \mathbb{R}$, and $V^i(t)$ are the components of a vector field along curve c such that $V^i(0) = V^i(1) = 0$. We look for necessary conditions for the curve $c = c_0$ to be an extremal of the integral

$$I(c_\varepsilon) = \int_0^1 L\left(x^i(t), \frac{\mathrm{d}x^i}{\mathrm{d}t}(t), \dots, \frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}x^i}{\mathrm{d}t^{k-1}} + \varepsilon V^i(t), \frac{1}{k!}\frac{\mathrm{d}^k x^i}{\mathrm{d}t^k}(t) + \varepsilon\frac{\mathrm{d}V^i}{\mathrm{d}t}(t)\right)\mathrm{d}t.$$

For this we require that c is a solution of the following equation:

$$0=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big(I(c_\varepsilon)\big)\Big|_{\varepsilon=0}=\int_0^1\left(\frac{\partial L}{\partial y^{(k-1)i}}V^i+\frac{\partial L}{\partial y^{(k)i}}\frac{\mathrm{d}V^i}{\mathrm{d}t}\right)\mathrm{d}t$$

$$=\int_0^1\left[\frac{\partial L}{\partial y^{(k-1)i}}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^{(2)i}}\right)\right]V^i\,\mathrm{d}t+\left(\frac{\partial L}{\partial y^{(2)i}}V^i\right)\Big|_{t=0}^{t=1}. \tag{4}$$

Since V is an arbitrary vector field, we have that Eq. (4) holds true if and only if the following equations are satisfied:

$$\frac{\partial L}{\partial y^{(k-1)i}}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^{(k)i}}\right)=0. \tag{5}$$

For a regular Lagrangian L, Eq. (5) represents a system of $k+1$ order differential equations that can be written as follows:

$$\frac{\mathrm{d}^{k+1}x^i}{\mathrm{d}t^{k+1}}+(k+1)!G^i\left(x,\frac{\mathrm{d}x}{\mathrm{d}t},\ldots,\frac{1}{k!}\frac{\mathrm{d}^kx}{\mathrm{d}t^k}\right)=0. \tag{6}$$

A geometric approach of such systems was proposed in [1]. Here functions G^i are given by

$$(k+1)G^j=\frac{1}{2}g^{ji}\left[\mathrm{d}_T\left(\frac{\partial L}{\partial y^{(k)i}}\right)-\frac{\partial L}{\partial y^{(k-1)i}}\right] \tag{7}$$

and they are local coefficients of a $k+1$ order vector field, [5].

Definition 2.1. The vector field S, given by expression (1) with coefficients G^i given by expression (7), is called the canonical semispray of the regular Lagrangian L.

In expression (7), d_T is the Tulczyjew operator, [7]

$$\mathrm{d}_T=y^{(1)i}\frac{\partial}{\partial x^i}+2y^{(2)i}\frac{\partial}{\partial y^{(1)i}}+\cdots+ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}}.$$

As it has been shown in [6], for a regular Lagrangian of order one, the canonical semispray is uniquely determined by the following equation $\mathcal{L}_S(\mathrm{d}_JL)=\mathrm{d}L$. The next theorem generalizes this result for a regular Lagrangian of order k

Theorem 2.2. *For a regular Lagrangian L of order k, the canonical semispray is the only semispray S on T^kM that satisfies the following equation*

$$\mathcal{L}_S(\mathrm{d}_{J^k}L)=\mathrm{d}_{J^{k-1}}L. \tag{8}$$

Proof. Consider S a semispray on T^kM, given by expression (1). The Lie derivative of the Cartan–Poincaré one-form $\theta_L^k=d_{J^k}L$ has the following expression

$$\mathcal{L}_S\theta_L^k=\left[\mathrm{d}_T\left(\frac{\partial L}{\partial y^{(k)i}}\right)-2(k+1)g_{ji}G^j\right]\mathrm{d}x^i+\frac{\partial L}{\partial y^{(k)i}}\,\mathrm{d}y^{(1)i}. \tag{9}$$

Using expression (9), we obtain that for a regular Lagrangian, Eq. (8) uniquely determines the functions G^i, which are given by expression (7). □

Corollary 2.3. *For a regular Lagrangian L of order k, its canonical semispray is uniquely determined by the following equation*

$$i_S\omega_L^k=-\mathrm{d}\big(\Gamma_1(L)\big)+\theta_L^k. \tag{10}$$

Proof. From Proposition 1.1 for $\alpha=k$ we have that $\mathcal{L}_S\theta_L^k=i_S\omega_L^k+d(\Gamma_1(L))$. Therefore, Eqs. (8) and (10) are equivalent and each of them, uniquely determine the canonical semispray of the Lagrange space. □

For a first order Lagrangian, the right hand side of expression (10) is an exact form, being the differential of the energy function. For higher order Lagrangians this is not true anymore, since θ_L^k is not even a closed one-form. This has the following consequences, which are different from the case of a first order Lagrangian.

Corollary 2.4. *For a regular Lagrangian of order k, its canonical semispray satisfies*:

(i) $\mathcal{L}_S\omega_L^k = \omega_L^{k-1}$,
(ii) $S(\Gamma_1(L)) = \Gamma_2(L)$.

Proof. If we take the exterior differential of both sides of Eq. (8) and use the fact that the Lie derivative commutes with the exterior differential, we obtain $\mathcal{L}_S\omega_L^k = \omega_L^{k-1}$. Using the skew symmetry of ω_L^k and expression (10), we obtain $S(\Gamma_1(L)) = \Gamma_2(L)$. □

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