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Dynamical Systems/Ordinary Differential Equations

Birth control in generalized Hopf bifurcations

Marc Chaperon

Institut de mathématiques de Jussieu & Université Paris 7, UFR de mathématiques, case 7012, 2, place Jussieu, 75251 Paris cedex 05, France

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Abstract

We state a very general lemma ensuring, at partially elliptic rest points of families of vector fields or transformations, the birth of normally hyperbolic invariant compact manifolds. A few examples follow. ***To cite this article: M. Chaperon, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Contrôle des naissances dans les bifurcations de Hopf généralisées. Nous énonçons un lemme très général garantissant, aux points stationnaires partiellement elliptiques de familles de champs de vecteurs ou de transformations, la naissance de variétés compactes invariantes normalement hyperboliques. Quelques exemples suivent. ***Pour citer cet article : M. Chaperon, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

Cette Note annonce un travail [2] fondé sur la ferme conviction que, dans le couplage d'oscillateurs, apparaissent d'autres phénomènes intéressants que des mouvements quasi-périodiques.

Hypothèses dans le cas des champs de vecteurs

Soit $(u,x)\mapsto Z_u(x)\in\mathbf{R}^m$ une famille assez différentiable de champs de vecteurs à paramètre $u\in\mathbf{R}^k$, définie au voisinage d'un point de $\mathbf{R}^k\times\mathbf{R}^m$ que l'on peut supposer être 0, telle que $Z_0(0)=0$ et que les valeurs propres de $DZ_0(0)$ soient imaginaires pures, simples et différentes de 0, d'où $m=2n$.

Les valeurs propres de partie imaginaire positive de $DZ_0(0)$ étant notées $\mathrm{i}\beta_1,\dots,\mathrm{i}\beta_n$, on suppose que, pour $1\leqslant j\leqslant n$, l'équation $\beta_j=\sum_1^n(p_\ell-q_\ell)\beta_\ell$ avec $(p,q)\in(\mathbf{N}^n)^2$ et $\sum(p_\ell+q_\ell)\leqslant 4$ n'a que les solutions évidentes $p_j=q_j+1$ et $p_\ell=q_\ell$ pour $\ell\neq j$.

Un changement $(u,x)\mapsto(u,g_u(x))$ de coordonnées locales permet de supposer que $Z_u(0)\equiv 0$, que $\mathbf{R}^m=\mathbf{C}^n$, que $DZ_0(0)$ est de la forme $z\mapsto(\mathrm{i}\beta_1z_1,\dots,\mathrm{i}\beta_nz_n)$ et que $Z_u=N_u+R_u$, où R_u s'annule à l'ordre 4 en 0 et N_u est

E-mail address: chaperon@math.jussieu.fr.

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donnée par (1). Dans cette formule, $z=(z_1,\dots,z_n)\in\mathbf{C}^n$ et $\lambda_j,\mu_j,a_{j\ell},b_{j\ell}$ sont des fonctions réelles différentiables telles que $\lambda_j(0)=0$ et $\mu_j(0)=\beta_j$. Il en résulte que l'application ρ de $\mathbf{C}^n$ sur $\mathbf{R}^n_+$ définie par $\rho(z)=(|z_1|,\dots,|z_n|)$ envoie N_u sur $\sum_j(\lambda_j(u)-\sum_\ell a_{j\ell}(u)r_\ell^2)r_j\frac{\partial}{\partial r_j}$.

À chaque $v\in\mathbf{R}^k$ on associe le champ de vecteurs $X_v(x)=\sum_j x_j(D\lambda_j(0)v-\sum_\ell a_{j\ell}(0)x_\ell^2)\frac{\partial}{\partial x_j}$ sur $\mathbf{R}^n$. Il est invariant sous l'action de $\mathbf{O}(1)^n$ engendrée par les symétries par rapport aux hyperplans de coordonnées. On suppose $v_0\in\mathbf{S}^{k-1}$ tel que $X_{v_0}(x)$ admette une variété invariante compacte Σ_{v_0} normalement hyperbolique [4,6], invariante par $\mathbf{O}(1)^n$ et dont l'intersection avec l'orthant positif $\mathbf{R}^n_+$ est connexe, d'où la propriété suivante :

(P) Si $\mathbf{R}^J$, $J\subset\{1,\dots,n\}$, est le plus petit sous-espace de coordonnées contenant Σ_{v_0}, alors Σ_{v_0} est transverse *dans* $\mathbf{R}^J$ aux sous-espaces de coordonnées contenus dans $\mathbf{R}^J$.

Par stabilité de l'hyperbolicité normale, il existe un voisinage ouvert V_{v_0} de v_0 dans $\mathbf{S}^{k-1}$ tel que chaque X_v avec $v\in V_{v_0}$ ait une variété invariante normalement hyperbolique Σ_v difféomorphe à Σ_{v_0} et proche de celle-ci, unique, donc $\mathbf{O}(1)^n$-invariante et possédant la propriété (P) avec le même J. Il en résulte que $S_{0,v}:=\rho^{-1}(\Sigma_v)\subset\mathbf{C}^J$ est une sous-variété compacte $\mathbf{U}(1)^n$-invariante, de mêmes différentiabilité et codimension que Σ_v.

Théorème 1 *(lemme de naissance). Sous ces hypothèses, il existe un ouvert* U_{v_0} *de* $\mathbf{R}^k$ *adhérent à* 0 *dont le cône tangent*[1] *en* 0 *est un cône ouvert de sommet* 0 *contenant* $\mathbf{R}^*_+V_{v_0}$, *tel que chacun des* Z_u *avec* $u\in U_{v_0}$ *admette une variété invariante compacte normalement hyperbolique* S_u *difféomorphe à* S_{0,v_0}, *de mêmes indice et co-indice que la variété invariante* Σ_{v_0} *de* X_{v_0}, *dépendant de manière au moins* $C^{1+\alpha}$ *de* u *et tendant vers* $\{0\}$ *quand* $u\to 0$. *Plus précisément, pour tout chemin lisse* $\gamma:(\mathbf{R}_+,0)\to(\mathbf{R}^k,0)$ *vérifiant* $\dot\gamma(0)\in V_{v_0}$, *on a* $\gamma(\varepsilon)\in U_{v_0}$ *pour* $\varepsilon>0$ *assez petit et* $\lim_{\varepsilon\to0}\varepsilon^{-1/2}S_{\gamma(\varepsilon)}=S_{0,\dot\gamma(0)}$ *au sens au moins* C^1.

Exemple 0. Si $J=\emptyset$ et donc $\Sigma_{v_0}=\{0\}$, l'hypothèse signifie que les $D\lambda_j(0)v_0$ sont tous non nuls et les $S_u=\{0\}$ sont des zéros hyperboliques.

Exemple 1 *(bifurcations de Hopf).* Pour $J=\{\ell\}$, l'hypothèse signifie que l'on a $D\lambda_j(0)v_0\neq0$ pour $j\neq\ell$, $a_{\ell\ell}(0)D\lambda_\ell(0)v_0>0$, $\Sigma_{v_0}=\{x\in\mathbf{R}^J:\ |x_\ell|=\sqrt{\frac{D\lambda_\ell(0)v_0}{a_{\ell\ell}(0)}}\}$ et $S_{0,v_0}=\{z\in\mathbf{C}^J:\ |z_\ell|=\sqrt{\frac{D\lambda_\ell(0)v_0}{a_{\ell\ell}(0)}}\}$. Les S_u sont donc des orbites périodiques et l'on obtient une version un peu faible de la bifurcation de Hopf.

Exemple 2 *(naissance de tores invariants de dimension $d\leqslant n$).* Pour $\#J=d>1$, l'hypothèse est vérifiée avec $\dim\Sigma_{v_0}=0$ si et seulement si l'on a les trois conditions suivantes : la matrice $A=(a_{j\ell})_{j,\ell\in J}$ est inversible, le point $y_{v_0}:=A^{-1}(D\lambda_j(0)v_0)_{j\in J}\in\mathbf{R}^J$ est à coordonnées strictement positives et le zéro $x_{v_0}\in\mathbf{R}^J$ de X_{v_0} défini par $x_{v_0j}=\sqrt{y_{v_0j}}$, $j\in J$, est hyperbolique. Dans ce cas, $\Sigma_{v_0}=\mathbf{O}(1)^nx_{v_0}$ et $S_{0,v_0}=\rho^{-1}(x_{v_0})$ est un tore $\mathbf{T}^d$ ainsi (donc) que les S_u.

Exemple 3 *(naissance de sphères invariantes de dimension $2d-1$, $2\leqslant d\leqslant n$).* Pour $\#J=d>1$, il peut arriver [7,8,2] que Σ_{v_0} soit une sphère $\mathbf{S}^{d-1}$ entourant l'origine dans $\mathbf{R}^J$, ce *de manière structurellement stable pour $k\geqslant n$* (et même $k\geqslant d$) [2]. Les S_u sont alors des sphères $\mathbf{S}^{2d-1}$. Si $d=n$, elles entourent l'origine dans $\mathbf{C}^n$ et le résultat ressemble plus à la bifurcation de Hopf que la naissance de tores invariants.

Exemple 4 *(naissance d'autres variétés de López de Medrano–Meersseman [9,1] invariantes).* Le lemme précédent permet (entre autres) de traiter facilement les cas où Mathilde Kammerer-Colin de Verdière avait mis en évidence la naissance de produits de sphères [7] et ceux où nous avions mis en évidence avec elle la naissance de sommes connexes de tels produits [3].

Remarque. Ces exemples ne s'excluent pas mutuellement. Ils seront développés dans [2], qui contiendra aussi la preuve du lemme de naissance. Celui-ci vaut pour les applications : voir ci-après.

[1] Ensemble des vitesses $\dot\gamma(0)$ de chemins dérivables $\gamma:(\mathbf{R},0)\to(\mathbf{R}^k,0)$ vérifiant $\gamma(\varepsilon)\in U_{v_0}$ pour $\varepsilon>0$.

This is an account of work [2] based on the firm belief that, when coupling oscillators, interesting phenomena occur beyond quasiperiodic motions.

1. The birth lemma for vector fields

Hypotheses. Let $(u,x)\mapsto Z_u(x)\in\mathbf{R}^m$ be a smooth enough family of vector fields with parameter $u\in\mathbf{R}^k$, defined in the neighbourhood of a point of $\mathbf{R}^k\times\mathbf{R}^m$ which we may assume to be 0, such that $Z_0(0)=0$ and that the eigenvalues of $DZ_0(0)$ are pure imaginary, simple and different from 0, hence $m=2n$.

Denoting by $\mathrm{i}\beta_1,\dots,\mathrm{i}\beta_n$ the eigenvalues with positive imaginary part, we assume that, for $1\leqslant j\leqslant n$, the equation $\beta_j=\sum_1^n(p_\ell-q_\ell)\beta_\ell$ with $(p,q)\in(\mathbf{N}^n)^2$ and $\sum(p_\ell+q_\ell)\leqslant 4$ admits only the obvious solutions $p_j=q_j+1$ and $p_\ell=q_\ell$ for $\ell\neq j$.

Via a local change of coordinates $(u,x)\mapsto(u,g_u(x))$, we can suppose that $\mathbf{R}^m=\mathbf{C}^n$, $Z_u(0)\equiv 0$, that $L:=DZ_0(0)$ is of the form $Lz=(\mathrm{i}\beta_1z_1,\dots,\mathrm{i}\beta_nz_n)$ and that $Z_u=N_u+R_u$, where R_u vanishes to order 4 at the origin and N_u is a (real) polynomial vector field of degree 3 on $\mathbf{C}^n$ which commutes with L and therefore (by the above hypothesis) is invariant under the natural action of $\mathbf{U}(1)^n$. In other words,

$$N_u(z)=\left(z_j\left(\lambda_j(u)+\mathrm{i}\mu_j(u)-\sum_{\ell=1}^n\big(a_{j\ell}(u)+\mathrm{i}b_{j\ell}(u)\big)|z_\ell|^2\right)\right)_{1\leqslant j\leqslant n},\tag{1}$$

where $z=(z_1,\dots,z_n)\in\mathbf{C}^n$ and $\lambda_j,\mu_j,a_{j\ell},b_{j\ell}$ are differentiable real functions with $\lambda_j(0)=0$ and $\mu_j(0)=\beta_j$. In particular, the mapping $\rho:\mathbf{C}^n\to\mathbf{R}^n_+$ defined by $\rho(z)=(|z_1|,\dots,|z_n|)$ sends N_u onto the vector field $\sum_j(\lambda_j(u)-\sum_\ell a_{j\ell}(u)r_\ell^2)r_j\frac{\partial}{\partial r_j}$.

To each $v\in\mathbf{R}^k$ we associate the vector field $X_v(x)=\sum_j x_j(D\lambda_j(0)v-\sum_\ell a_{j\ell}(0)x_\ell^2)\frac{\partial}{\partial x_j}$ on $\mathbf{R}^n$. It is invariant under the action of $\mathbf{O}(1)^n$ generated by the symmetries with respect to coordinate hyperplanes. We assume $v_0\in\mathbf{S}^{k-1}$ such that $X_{v_0}(x)$ admits a compact normally hyperbolic [4,6] invariant manifold Σ_{v_0}, invariant by $\mathbf{O}(1)^n$ and whose intersection with the non-negative orthant $\mathbf{R}^n_+$ is connected, hence the following property:

(P) If $\mathbf{R}^J$, $J\subset\{1,\dots,n\}$, is the smallest coordinate subspace which contains Σ_{v_0}, then Σ_{v_0} is transversal *in* $\mathbf{R}^J$ to the coordinate subspaces lying in $\mathbf{R}^J$.

As normal hyperbolicity is open, there exists an open neighbourhood V_{v_0} of v_0 in $\mathbf{S}^{k-1}$ such that every X_v with $v\in V_{v_0}$ has a normally hyperbolic invariant manifold Σ_v diffeomorphic to Σ_{v_0} and close to it, unique and therefore $\mathbf{O}(1)^n$-invariant, such that Σ_v satisfies (P) with the same J. It follows that $S_{0,v}:=\rho^{-1}(\Sigma_v)\subset\mathbf{C}^J$ is a compact $\mathbf{U}(1)^n$-invariant submanifold with the same smoothness and codimension as Σ_v.

Remarks. For positive ε, the submanifold $\varepsilon^{1/2}\Sigma_{v_0}$ is a normally hyperbolic invariant manifold of $X_{\varepsilon v_0}$.Thus, if the mapping $u\mapsto(\lambda_1(u),\dots,\lambda_n(u))$ is a submersion at 0 (which is generically the case if $k=n$), the existence of Σ_{v_0} is equivalent to that of $\nu\in\mathbf{R}^n$ such that the vector field $\sum_j x_j(\nu_j-\sum_\ell a_{j\ell}(0)x_\ell^2)\frac{\partial}{\partial x_j}$ has a compact normally hyperbolic invariant manifold as before.

Using diagonal changes of coordinates in $\mathbf{C}^n$, one can see that, for each ℓ, the functions $a_{j\ell}+\mathrm{i}b_{j\ell}$ are defined up to multiplication by the same positive function of u.

Theorem 1 *(birth lemma). Under those hypotheses, there exists an open subset U_{v_0} of $\mathbf{R}^k$, whose closure contains* 0 *and whose tangent cone*[2] *at* 0 *is an open cone with vertex* 0 *containing $\mathbf{R}^*_+V_{v_0}$, such that every Z_u with $u\in U_{v_0}$ has a compact normally hyperbolic invariant manifold S_u diffeomorphic to S_{0,v_0}, with the same index and co-index as the invariant manifold Σ_{v_0} of X_{v_0}, depending at least $C^{1+\alpha}$ on u and tending to $\{0\}$ when $u\to 0$. More precisely, for each smooth path $\gamma:(\mathbf{R}_+,0)\to(\mathbf{R}^k,0)$ with $\dot\gamma(0)\in V_{v_0}$, one has $\gamma(\varepsilon)\in U_{v_0}$ for small enough positive ε and $\lim_{\varepsilon\to 0}\varepsilon^{-1/2}S_{\gamma(\varepsilon)}=S_{0,\dot\gamma(0)}$ in the at least C^1 sense.*

[2] Set of all velocities $\dot\gamma(0)$ of differentiable paths $\gamma:(\mathbf{R},0)\to(\mathbf{R}^k,0)$ satisfying $\gamma(\varepsilon)\in U_{v_0}$ for all positive ε.

Example 1.0. If $J=\emptyset$ and therefore $\Sigma_{v_0}=\{0\}$, the hypothesis means that every $D\lambda_j(0)v_0$ is non-zero, and the lemma does select values of u for which $S_u=\{0\}$ is a hyperbolic rest point of Z_u.

Example 1.1 *(Hopf bifurcations).* If $J=\{\ell\}$, the hypothesis means that one has $a_{\ell\ell}(0)D\lambda_\ell(0)v_0>0$, $D\lambda_j(0)v_0\neq 0$ for $j\neq\ell$, $\Sigma_{v_0}=\{x\in\mathbf{R}^J: |x_\ell|=\sqrt{\frac{D\lambda_\ell(0)v_0}{a_{\ell\ell}(0)}}\}$ and $S_{0,v_0}=\{z\in\mathbf{C}^J: |z_\ell|=\sqrt{\frac{D\lambda_\ell(0)v_0}{a_{\ell\ell}(0)}}\}$. Therefore, the S_u's are periodic orbits and we get a somewhat weak version of the Hopf bifurcation.

Example 1.2 *(birth of invariant tori of dimension $d\leqslant n$).* For $\#J=d>1$, the hypothesis is satisfied with $\dim\Sigma_{v_0}=0$ if and only if the following three conditions hold: the matrix $A=(a_{j\ell})_{j,\ell\in J}$ is invertible, the point $y_{v_0}:=A^{-1}(D\lambda_j(0)v_0)_{j\in J}\in\mathbf{R}^J$ has all its coordinates positive and the zero $x_{v_0}\in\mathbf{R}^J$ of X_{v_0} defined by $x_{v_0 j}=\sqrt{y_{v_0 j}}$, $j\in J$, is hyperbolic. Then, $\Sigma_{v_0}=\mathbf{O}(1)^n x_{v_0}$ and $S_{0,v_0}=\rho^{-1}(x_{v_0})$ is a d-dimensional torus, hence so are the S_u's.

Example 1.3 *(birth of invariant spheres of dimension $2d-1$, $2\leqslant d\leqslant n$).* For $\#J=d>1$, the submanifold Σ_{v_0} can be [7,8,2] a sphere $\mathbf{S}^{d-1}$ around the origin in $\mathbf{R}^J$, a situation *which can be structurally stable for $k\geqslant n$* (in fact, $k\geqslant d$) [2]. Then, the S_u's are spheres $\mathbf{S}^{2d-1}$. If $d=n$, they lie around the origin in $\mathbf{C}^n$ and the result is much more reminiscent of the Hopf bifurcation than the birth of invariant tori.

In contrast with the latter case, the existence of such a sphere Σ_{v_0} is non-trivial. To introduce more familiar objects, one can remark that the change of variables $y=\pi(x):=(x_1^2,\dots,x_n^2)$ sends X_{v_0} onto the Lotka–Volterra field $Y_{v_0}(y)=2\sum_j y_j(D\lambda_j(0)v_0-\sum_\ell a_{j\ell}(u_0)y_\ell)\frac{\partial}{\partial y_j}$ on $\mathbf{R}^n_+$. The sphere Σ_{v_0} is simply the inverse image by π of the carrying simplex σ_{v_0} of Y_{v_0}, whose existence can follow from Hirsch's theorem [5]. For example, if σ_{v_0} is the standard simplex $\sum y_j=1$ of $\mathbf{R}^n_+$, then $\Sigma_{v_0}=\pi^{-1}(\sigma_{v_0})$ is the unit sphere $\sum x_j^2=1$ of $\mathbf{R}^n$ and S_{0,v_0} is the unit sphere $\sum|z_j|^2=1$ of $\mathbf{C}^n$. Details will be given in [2].

Example 1.4 *(birth of other invariant López de Medrano–Meersseman manifolds [9,1]).* The birth lemma enables one (among other things) to treat easily the cases where Mathilde Kammerer-Colin de Verdière had established the birth of products of spheres [7] and those where we had established together the birth of connected sums of such products [3].

Note that those examples do not exclude each other.

2. The birth lemma for maps

Hypotheses. Let $\mathbf{R}^k\times\mathbf{R}^m\ni(u,x)\mapsto h_u(x)\in\mathbf{R}^m$ be a smooth enough family of maps, defined in the neighbourhood of a point of $\mathbf{R}^k\times\mathbf{R}^m$ which we may assume to be 0, such that $h_0(0)=0$ and that the eigenvalues of $Dh_0(0)$ are of modulus 1, simple and different from 1, hence either $m=2n$ with no real eigenvalue, or $m=2n-1$ with the sole real eigenvalue -1.

Denoting the eigenvalues by $\mathrm{e}^{\pm\mathrm{i}\beta_1},\dots,\mathrm{e}^{\pm\mathrm{i}\beta_n}$, $0<\beta_1<\dots<\beta_n\leqslant\pi$, we assume that, for $1\leqslant j\leqslant n$, the equation $\mathrm{e}^{\mathrm{i}\beta_j}=\prod_1^n\mathrm{e}^{\mathrm{i}(p_\ell-q_\ell)\beta_\ell}$ with $(p,q)\in(\mathbf{N}^n)^2$ and $\sum(p_\ell+q_\ell)\leqslant 4$ admits only the obvious solutions $p_j=q_j+1$ and $p_\ell=q_\ell$ for $\ell\neq j$.

Via a local change of coordinates $(u,x)\mapsto(u,g_u(x))$, we can suppose that $\mathbf{R}^m=\mathbf{C}^n$ or $\mathbf{C}^{n-1}\times\mathbf{R}$, $h_u(0)\equiv 0$, that $Dh_0(0)$ is the map $z\mapsto(\mathrm{e}^{\mathrm{i}\beta_1}z_1,\dots,\mathrm{e}^{\mathrm{i}\beta_n}z_n)$ and that $h_u=\sigma\circ g^1_{N_u}+R_u$, where R_u vanishes to order 4 at the origin,

$$\sigma(z)=\begin{cases} z & \text{if } m=2n,\\ (z_1,\dots,z_{n-1},-z_n) & \text{if } m=2n-1,\end{cases}$$

and $g^t_{N_u}$ is the flow of a vector field N_u of the form (1) with $\lambda_j(0)=0$ and

$$\begin{cases} \mu_j(0)=\beta_j & \text{if } m=2n,\\ \mu_j(0)=\beta_j & \text{for } j<n \text{ and (of course) } \mu_n(u)=b_{n\ell}(u)=0 \text{ if } m=2n-1.\end{cases}$$

As in Section 1, we associate to N_u the vector fields X_v on $\mathbf{R}^n$, and we assume that X_{v_0} has a compact normally hyperbolic invariant manifold Σ_{v_0}, invariant by $\mathbf{O}(1)^n$ and such that $\Sigma_{v_0}\cap\mathbf{R}^n_+$ is connected. With the notation of Section 1, we have the same result and similar examples:

Theorem 2 *(birth lemma). Under those hypotheses, there exists an open subset* U_{v_0} *of* $\mathbf{R}^k$*, whose closure contains* 0 *and whose tangent cone at* 0 *is an open cone with vertex* 0 *containing* $\mathbf{R}_+^* V_{v_0}$*, such that every* h_u *with* $u \in U_{v_0}$ *has a compact normally hyperbolic invariant manifold* S_u *diffeomorphic to* S_{0,v_0}*, with the same index and co-index as the invariant manifold* Σ_{v_0} *of* X_{v_0}*, depending at least* $C^{1+\alpha}$ *on* u *and tending to* $\{0\}$ *when* $u \to 0$*. More precisely, for each smooth path* $\gamma : (\mathbf{R}_+, 0) \to (\mathbf{R}^k, 0)$ *with* $\dot\gamma(0) \in V_{v_0}$*, one has* $\gamma(\varepsilon) \in U_{v_0}$ *for small enough positive* ε *and* $\lim_{\varepsilon \to 0} \varepsilon^{-1/2} S_{\gamma(\varepsilon)} = S_{0,\dot\gamma(0)}$ *in the at least* C^1 *sense.*

Example 2.0. If $J = \emptyset$, the lemma selects values of u for which $S_u = \{0\}$ is a hyperbolic rest point of h_u.

Example 2.1 *(Poincaré–Andronov and "Hopf" bifurcations).* Of course, if $\#J = 1$, the hypothesis has the same meaning for X_{v_0} as in example 1.1.

If $J = \{n\}$, $m = 2n - 1$, then $S_{0,v_0} = \Sigma_{v_0} = \{z \in \mathbf{C}^{n-1} \times \mathbf{R}\colon z_1 = \cdots = z_{n-1} = 0, |z_n| = \sqrt{\frac{D\lambda_n(0)v_0}{a_{nn}(0)}}\}$. Therefore, the S_u's are 2-periodic orbits and we get a (bad) version of the period doubling bifurcation.

If $J = \{\ell\}$ (with $\ell < n$ if $m = 2n - 1$), the hypothesis implies that $S_{0,v_0} = \{z \in \mathbf{C}^J\colon |z_\ell| = \sqrt{\frac{D\lambda_\ell(0)v_0}{a_{\ell\ell}(0)}}\}$. Therefore, the S_u's are invariant closed curves and we get the "Hopf" bifurcation for maps.

Example 2.2 *(birth of invariant tori of dimension* $d \leqslant n$*).* For $\#J = d > 1$ and $\dim \Sigma_{v_0} = 0$, as in Example 1.2, the hypothesis implies that $\Sigma_{v_0} = \mathbf{O}(1)^n x_{v_0}$. It follows that the S_u's are d-dimensional tori except if $m = 2n - 1$ and $n \in J$, in which case they consist of a pair of $(d-1)$-tori exchanged by h_u.

Example 2.3 *(birth of invariant spheres).* For $\#J = d > 1$, if Σ_{v_0} is a sphere $\mathbf{S}^{d-1}$ around the origin in $\mathbf{R}^J$, then the S_u's are $(2d-1)$-spheres except for $m = 2n - 1$ and $n \in J$, in which case they are $(2d-2)$-spheres. If $d = n$, we get a very natural generalization of the Hopf bifurcation for maps.

Example 2.4 *(birth of other invariant López de Medrano–Meersseman manifolds).* As for vector fields, the birth lemma enables one (among other things) to establish quite easily the birth of products of spheres [7,8] and that of connected sums of such products [3,8].

Again, those examples do not exclude each other. The advantage of maps is that for $m = 3$ one can draw pictures. The content of the present Note will be developed in the forthcoming paper [2].

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