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Topology/Group Theory

The Casson invariant and the word metric on the Torelli group

Nathan Broaddus [1], Benson Farb [1], Andrew Putman

Department of Mathematics, University of Chicago, 5734 S. University Avenue, Chicago, IL 60637, USA

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Abstract

We bound the value of the Casson invariant of any integral homology 3-sphere M by a constant times the distance-squared to the identity, measured in any word metric on the Torelli group $\mathcal{I}$, of the element of $\mathcal{I}$ associated to any Heegaard splitting of M. We construct examples which show this bound is asymptotically sharp. ***To cite this article: N. Broaddus et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

L'invariant de Casson et la métrique des mots sur le groupe de Torelli. Soit M une sphère d'homologie de dimension 3. Tout scindement de Heegaard de M définit un élément du groupe de Torelli $\mathcal{I}$. Nous montrons que l'invariant de Casson de M est borné par une constante fois le carré de la longueur de cet élément. Cette longueur est définie comme la longueur minimale d'un mot le représentant, écrit en utilisant un système générateur fini quelconque de $\mathcal{I}$. Nous construisons des exemples qui montrent que cette borne est asymptotiquement la meilleure possible. ***Pour citer cet article : N. Broaddus et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

The *Casson invariant* $\lambda(M) \in \mathbb{Z}$ is a fundamental and well-studied invariant of integral homology 3-spheres M. Roughly speaking, $\lambda(M)$ is half the algebraic number of conjugacy classes of irreducible representations of $\pi_1(M)$ into $\mathrm{SU}(2)$. See [1] for a thorough exposition of the Casson invariant.

The *mapping class group* Mod_g of a closed, orientable, genus g surface Σ_g is the group of homotopy classes of orientation-preserving homeomorphisms of Σ_g. The subgroup of Mod_g consisting of elements acting trivially on $\mathrm{H}_1(\Sigma_g;\mathbb{Z})$ is called the *Torelli group*, and is denoted by $\mathcal{I}_g$.

Let M be an integral homology 3-sphere, and let $f:\Sigma_g \to M$ be a Heegaard embedding. For any $\phi \in \mathcal{I}_g$, denote by M_ϕ the homology 3-sphere obtained by cutting M along $f(\Sigma_g)$ and gluing back the resulting two handlebodies

E-mail addresses: broaddus@math.uchicago.edu (N. Broaddus), farb@math.uchicago.edu (B. Farb), andyp@math.uchicago.edu (A. Putman).

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M^+ and M^- along their boundaries via the homeomorphism ϕ. Note that any integral homology 3-sphere can be obtained from $M = S^3$ in this way.

In this Note we give a sharp asymptotic bound on $|\lambda(M_\phi)|$ in terms of the word metric on $\mathcal{I}_g$. To explain our result, we fix $g > 2$ and pick once and for all a finite set S of generators for $\mathcal{I}_g$; the fact that $\mathcal{I}_g$ is finitely generated when $g > 2$ is a deep result of D. Johnson (see [3]). Denote by $\|\cdot\|$ the induced word norm on $\mathcal{I}_g$; i.e. $\|\phi\|$ is the length of the shortest word in $S^{\pm 1}$ which equals ϕ. Different choices of finite generating sets for $\mathcal{I}_g$ give word norms whose ratios are bounded by a constant. For a fixed Heegaard embedding $f : \Sigma_g \to M$, Morita [5] has defined a kind of *normalized Casson invariant* $\lambda_f : \mathcal{I}_g \to \mathbb{Z}$ via

$$\lambda_f(\phi) := \lambda(M_\phi) - \lambda(M).$$

In particular, if $M = S^3$ and $h : \Sigma_g \to S^3$ is the unique genus g Heegaard embedding then $\lambda(S^3) = 0$, so the normalized Casson invariant λ_h satisfies $\lambda_h(\phi) = \lambda(S^3_\phi)$.

Theorem 1. *Let M be an oriented integral homology 3-sphere, let $g > 2$, and let $f : \Sigma_g \to M$ be a Heegaard embedding. Then there exists a constant $C > 0$ so that $|\lambda_f(\phi)| \leqslant C\|\phi\|^2$ for every $\phi \in \mathcal{I}_g$. This bound is sharp in the sense that there exists an infinite set $\{\phi_n\} \subset \mathcal{I}_g$ and a constant $K > 0$ so that $|\lambda_f(\phi_n)| \geqslant K\|\phi_n\|^2$ for all n.*

For the case $g = 2$, the Torelli group $\mathcal{I}_2$ is not finitely generated [4].

2. Morita's formula

Our proof of Theorem 1 relies in an essential way on a beautiful formula due to Morita [5] for $\lambda_f(\phi)$, which we now explain (following §4 of [5]). This formula measures the extent to which λ_f fails to be a homomorphism. This failure is encoded as a function $\delta_f : \mathcal{I}_g \times \mathcal{I}_g \to \mathbb{Z}$ defined as follows. Let $\mathcal{I}_{g,1}$ denote the Torelli group of an oriented, genus g surface with one boundary component $\Sigma_{g,1}$. In other words, $\mathcal{I}_{g,1}$ is the group of homotopy classes of orientation-preserving homeomorphisms of $\Sigma_{g,1}$ which fix the boundary pointwise, modulo homotopies which do the same and where the homeomorphisms act trivially on $H := \mathrm{H}_1(\Sigma_g; \mathbb{Z})$. Gluing a disc to $\partial\Sigma_{g,1}$ induces a natural surjective homomorphism $\pi : \mathcal{I}_{g,1} \to \mathcal{I}_g$, and there is a corresponding commutative diagram of *Johnson homomorphisms* (cf. [2] for discussions of these homomorphisms τ and their remarkable properties):

$$\begin{tikzcd}
\mathcal{I}_{g,1} \arrow[r, "\tau"] \arrow[d, "\pi"'] & \wedge^3 H \arrow[d] \\
\mathcal{I}_g \arrow[r, "\tau"] & \wedge^3 H/H.
\end{tikzcd}$$

The map $f : \Sigma_g \to M$ induces homomorphisms $H \to \mathrm{H}_1(M^\pm; \mathbb{Z})$ whose kernels we denote by H^+ and H^-, respectively. It is then easy to see that $H^+ \otimes \mathbb{R}$ and $H^- \otimes \mathbb{R}$ are maximal isotropic subspaces of the symplectic vector space $H \otimes \mathbb{R}$, and that

$$H = H^+ \oplus H^-.$$

Moreover, since M is an integral homology 3-sphere, there is a symplectic basis $\{x_1, \ldots, x_g, y_1, \ldots, y_g\}$ for H with $x_i \in H^+$ and $y_i \in H^-$. Now, given any two $\phi, \psi \in \mathcal{I}_g$, choose any lifts $\tilde{\phi}, \tilde{\psi}$ to $\mathcal{I}_{g,1}$. Using the obvious basis for $\wedge^3 H$ coming from our choice of basis for H, we can write

$$\tau(\tilde{\phi}) = \left[\sum_{i<j<k} a_{ijk}\, y_i \wedge y_j \wedge y_k\right] + \text{other terms},$$

$$\tau(\tilde{\psi}) = \left[\sum_{i<j<k} b_{ijk}\, x_i \wedge x_j \wedge x_k\right] + \text{other terms}$$

for some $a_{ijk}, b_{ijk} \in \mathbb{Z}$. Morita defines

$$\delta_f(\phi, \psi) = \sum_{i<j<k} a_{ijk} b_{ijk}$$

and proves that $\delta_f(\phi,\psi)$ does not depend on either the choice of lifts $\tilde{\phi}$, $\tilde{\psi}$ or the choice of symplectic basis for H. Morita then proves, as Theorem 4.3 of [5], that the following formula holds for all $\phi,\psi\in\mathcal{I}_g$:

$$\lambda_f(\phi\psi)=\lambda_f(\phi)+\lambda_f(\psi)+2\delta_f(\phi,\psi). \tag{1}$$

3. Proof of Theorem 1

Let $\{x_1,\dots,x_g,y_1,\dots,y_g\}$ be the standard basis for $H:=\mathrm{H}_1(\Sigma_g;\mathbb{Z})$ discussed in the previous section. For any vector $v\in\wedge^3H$, we denote by $\ell(v)$ the maximum of the absolute values of the coefficients of v with respect to the induced basis for $\wedge^3H$.

We want to relate $\lambda_f(\phi)$ to the word length of ϕ in $\mathcal{I}_g$, but Morita's formula (1) is computed using elements of $\mathcal{I}_{g,1}$, not of $\mathcal{I}_g$. To address this point, we first recall that gluing a disk to $\partial\Sigma_{g,1}$ induces an exact sequence

$$1\to\pi_1(T^1\Sigma_g)\to\mathcal{I}_{g,1}\xrightarrow{\pi}\mathcal{I}_g\to 1$$

where $T^1\Sigma_g$ is the unit tangent bundle of Σ_g. For each generator $s\in S$ of $\mathcal{I}_g$, choose a single lift $\tilde{s}\in\mathcal{I}_{g,1}$, and denote by $\widetilde{S}$ the union of these elements. We can then choose as a generating set for $\mathcal{I}_{g,1}$ the set $\widetilde{S}$ together with a finite generating set for $\pi_1(T^1\Sigma_g)$. With these choices of generating sets, we note that each $\phi\in\mathcal{I}_g$ has some lift $\tilde{\phi}$ so that

$$\|\tilde{\phi}\|_{\mathcal{I}_{g,1}}=\|\phi\|_{\mathcal{I}_g}. \tag{2}$$

This equality follows by writing out ϕ as a product of elements of S, then lifting generator by generator. Henceforth whenever we choose a lift of an element $\phi\in\mathcal{I}_g$, we will always choose a lift $\tilde{\phi}$ satisfying (2). The main point is that in computing with (1), we are allowed to choose any lifts, since Morita proves that $\delta_f(\phi,\psi)$ does not depend on the choice of lifts. Thus we can choose lifts which do not alter word length.

Now since $\widetilde{S}$ is finite, there exists C_1 so that

$$\ell\big(\tau(\tilde{s})\big)\leqslant C_1\quad\text{for all } s\in\widetilde{S}^{\pm1}. \tag{3}$$

Since τ is a homomorphism to the abelian group $\wedge^3H$, it follows from (3) that

$$\ell\big(\tau(\tilde{\phi})\big)\leqslant C_1\|\tilde{\phi}\|\quad\text{for all } \tilde{\phi}\in\mathcal{I}_{g,1}. \tag{4}$$

Finally, consider $\phi,\psi\in\mathcal{I}_g$ together with lifts $\tilde{\phi}$, $\tilde{\psi}$ satisfying (2). If a_{ijk} (resp. b_{ijk}) are the coordinates of $\tau(\tilde{\phi})$ (resp. $\tau(\tilde{\psi})$) as in the previous section, then

$$\big|\delta_f(\phi,\psi)\big|=\Bigg|\sum_{i<j<k}a_{ijk}b_{ijk}\Bigg|\leqslant\Bigg|\sum_{i<j<k}\ell\big(\tau(\tilde{\phi})\big)\ell\big(\tau(\tilde{\psi})\big)\Bigg|\leqslant\sum_{i<j<k}C_1^2\|\phi\|\|\psi\|\leqslant C_2\|\phi\|\|\psi\| \tag{5}$$

where $C_2=\binom{2g}{3}C_1^2$.

Now given any $\phi\in\mathcal{I}_g$, write $\phi=s_1\cdots s_n$, where each s_i is an element of $S^{\pm1}$ and where $n=\|\phi\|$. An iterated use of Morita's formula (1) gives

$$\begin{aligned}\lambda_f(\phi)&=\lambda_f(s_1)+\lambda_f(s_2\cdots s_n)+2\delta_f(s_1,s_2\cdots s_n)\\&=\lambda_f(s_1)+\lambda_f(s_2)+\lambda_f(s_3\cdots s_n)+2\delta_f(s_1,s_2\cdots s_n)+2\delta_f(s_2,s_3\cdots s_n)\\&\ \ \vdots\\&=\sum_{i=1}^{n}\lambda_f(s_n)+2\sum_{i=1}^{n-1}\delta_f(s_i,s_{i+1}\cdots s_n).\end{aligned} \tag{6}$$

Since S is finite, there exists $C_3>0$ so that $|\lambda_f(s)|\leqslant C_3$ for every $s\in S$. For some $C>0$, we thus have

$$\big|\lambda_f(\phi)\big|\leqslant\sum_{i=1}^{n}\big|\lambda_f(s_n)\big|+2\sum_{i=1}^{n-1}\big|\delta_f(s_i,s_{i+1}\cdots s_n)\big|\leqslant C_3n+2\sum_{i=1}^{n-1}C_2\cdot1\cdot(n-i)\leqslant Cn^2=C\|\phi\|^2.$$

The first claim of the theorem follows.

We now consider the second claim. Johnson proved (see, e.g. [2]) that the homomorphisms τ are surjective. Hence there exists some $\nu \in \mathcal{I}_g$ so that for some lift $\tilde{\nu} \in \mathcal{I}_{g,1}$ we have

$$\tau(\tilde{\nu}) = x_1 \wedge x_2 \wedge x_3 + y_1 \wedge y_2 \wedge y_3,$$

and hence

$$\tau(\tilde{\nu}^n) = n(x_1 \wedge x_2 \wedge x_3) + n(y_1 \wedge y_2 \wedge y_3). \tag{7}$$

Note that the choice of ν depends in a nontrivial way on the Heegaard embedding $f : \Sigma_g \to M$, so ν is not given explicitly. By Eq. (6), we have

$$\lambda_f(\nu^n) = \sum_{i=1}^{n} \lambda_f(\nu) + 2\sum_{i=1}^{n-1} \delta_f(\nu, \nu^{n-i}). \tag{8}$$

Now let $K_1 = |\lambda_f(\nu)|$, which is a constant since ν is fixed. By (7) and the definition of δ_f, we have for any $m > 0$ that $\delta_f(\nu, \nu^m) = m$. Thus by Eq. (8) there is some N such that for all $n \geqslant N$ we have

$$\left|\lambda_f(\nu^n)\right| = \left|\sum_{i=1}^{n} \lambda_f(\nu) + 2\sum_{i=1}^{n-1}(n-i)\right| \geqslant 2\sum_{i=1}^{n-1}(n-i) - \sum_{i=1}^{n} K_1 \geqslant K_2 n^2$$

for some $K_2 > 0$. If $\|\nu\| = K_3$, then clearly $\|\nu^n\| \leqslant K_3 n$. Thus

$$\left|\lambda_f(\nu^n)\right| \geqslant K_2 n^2 \geqslant \frac{K_2}{K_3^2}\|\nu^n\|^2 \quad \text{for all } n \geqslant N.$$

Setting $K = K_2/K_3^2$ we get the desired infinite set $\{\nu^n \mid n \geqslant N\} \subset \mathcal{I}_g$ establishing the asymptotic tightness of the upper bound.

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