Differential Geometry

The Ricci iteration and its applications

Yanir A. Rubinstein [*]

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Presented by Jean-Michel Bismut

Abstract

In this Note we introduce and study dynamical systems related to the Ricci operator on the space of Kähler metrics as discretizations of certain geometric flows. We pose a conjecture on their convergence towards canonical Kähler metrics and study the case where the first Chern class is negative, zero or positive. This construction has several applications in Kähler geometry, among them an answer to a question of Nadel and a construction of multiplier ideal sheaves. *To cite this article: Y.A. Rubinstein, C. R. Acad. Sci. Paris, Ser. I 345 (2007).*

Résumé

L'itération de Ricci et ses applications. Dans cette Note nous introduisons et étudions des systèmes dynamiques reliées à l'opérateur de Ricci sur l'espace des métriques kählériennes comme discrétisations des certains flots géométriques. Nous posons une conjecture concernant leurs convergence vers des métriques kählériennes canoniques et nous étudions le cas où la première classe de Chern est négative, zéro ou positive. Cette construction a plusieurs applications en géométrie kählérienne, parmi elles une réponse à une question de Nadel et une construction des faisceaux d'idéaux multiplicateurs. *Pour citer cet article : Y.A. Rubinstein, C. R. Acad. Sci. Paris, Ser. I 345 (2007).*

1. Introduction

Our main purpose in this Note is to describe a new method for the construction of canonical Kähler metrics via the discretization of certain geometric flows. The idea is to turn a geometric flow into a set of difference equations. Complete proofs will appear elsewhere [12].

The search for a canonical metric representative of a fixed Kähler class has been at the heart of Kähler geometry since its birth and has a long history starting from Kähler and continuing, among many others, with the work of Calabi, Aubin, Yau, Tian and Donaldson. For general extremal metrics a general existence theory is not presently available although the so-called Yau–Tian–Donaldson conjecture suggests that it should be related with notions of stability in algebraic geometry.

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[*] Present address: Department of Mathematics, Princeton University, Princeton, NJ 08544, USA.
E-mail addresses: yanir@member.ams.org, yanir@math.princeton.edu.

One of the main tools in the existence theory of Kähler–Einstein metrics and Kähler–Ricci solitons has been the Ricci flow introduced by Hamilton [6]. Cao has shown that Yau's continuity method proof may be phrased in terms of the convergence of the Ricci flow [3]. Much work has gone into understanding the Ricci flow on Fano manifolds and recently Perelman and Tian and Zhu proved that the analogous convergence result holds in this case [14]. The idea that there might be another way of approaching canonical metrics, in the form of a discrete iterative dynamical system, was suggested by Nadel [8] and is the main motivation for our work.

2. The Ricci iteration

Let (M, J, ω) be a connected compact closed Kähler manifold of complex dimension n and let Ω denote a Kähler class. Let $\Delta_\omega = -\bar\partial \circ \bar\partial^\star - \bar\partial^\star \circ \bar\partial$ denote the Laplacian with respect to ω. Let H_ω denote the Hodge projection operator from the space of closed forms onto the kernel of Δ_ω. Let $V = \Omega^n([M])$. Denote by $\mathcal{D}_\Omega$ the space of all closed $(1, 1)$-forms cohomologous to Ω, by $\mathcal{H}_\Omega$ the subspace of Kähler forms and by $\mathcal{H}_\Omega^+$ the subspace of Kähler forms whose Ricci curvature is positive. We introduce the following dynamical system on $\mathcal{H}_\Omega$ which is our main object of study in this Note:

Definition 2.1. Given a Kähler form $\omega \in \mathcal{H}_\Omega$ define the (time one) Ricci iteration[1] to be the sequence of forms $\{\omega_k\}_{k \geqslant 0}$, satisfying the following equations for each $k \in \mathbb{N}$ for which a solution exists

$$\omega_{k+1} = \omega_k + H_{k+1} \operatorname{Ric} \omega_{k+1} - \operatorname{Ric} \omega_{k+1}, \quad k + 1 \in \mathbb{N}, \quad \omega_0 = \omega.$$

This system of equations may be viewed as a discrete version of the flow $\frac{\partial \omega(t)}{\partial t} = -\operatorname{Ric} \omega(t) + H_t \operatorname{Ric} \omega(t)$. This flow, first studied by Guan [5], can in turn be considered as a Kähler version of Hamilton's Ricci flow. Our work is motivated by the following conjecture (an analogue may also be posed for the flow):

Conjecture 2.1. *Let (M, J) be a compact closed Kähler manifold, and assume that there exists a constant scalar curvature Kähler metric in $\mathcal{H}_\Omega$. Then for any $\omega \in \mathcal{H}_\Omega$ the Ricci iteration exists for all $k \in \mathbb{N}$ and converges in an appropriate sense to a constant scalar curvature metric.*

Our motivation for posing this conjecture comes from the following theorem:

Theorem 2.2. *Let (M, J) be a compact closed Kähler manifold admitting a Kähler–Einstein metric. Let Ω be a Kähler class such that $\mu\Omega = c_1$ with $\mu \in \{0, \pm 1\}$. Then for any $\omega \in \mathcal{H}_\Omega$ the Ricci iteration exists for all $k \in \mathbb{N}$ and converges in the sense of Cheeger–Gromov to a Kähler–Einstein metric.*

Remark 1. Note that the above conjecture may also be posed for solitons of the above flow, i.e., solutions of $\mathcal{L}_X \omega = \operatorname{Ric} \omega - H_\omega \operatorname{Ric} \omega$ for a holomorphic vector field X, and that a result analogous to Theorem 2.2 then holds for these metrics using the same methods and discretizing the flow twisted by the one-parameter subgroup of automorphisms corresponding to X.

Sketch of proof. First we prove that the iteration exists for each $k \in \mathbb{N}$. Since $H_\omega \operatorname{Ric} \omega = \mu\omega$, this amounts to solving $\omega_1 = \omega_0 + \mu\omega_1 - \operatorname{Ric} \omega_1$. Let $\omega_1 = \omega_{\varphi_1} := \omega + \sqrt{-1}\, \partial\bar\partial\varphi_1$. This can be written as a complex Monge–Ampère equation: $\omega_{\varphi_1}^n = \omega^n \mathrm{e}^{f_\omega - (\mu-1)\varphi_1}$, $\int_M \omega^n \mathrm{e}^{f_\omega - (\mu-1)\varphi_1} = V$, where $\sqrt{-1}\, \partial\bar\partial f_\omega = \operatorname{Ric} \omega - \omega$, $\int_M \mathrm{e}^{f_\omega} \omega^n = V$. The existence of solutions to such equations is known when $\mu \leqslant 0$ by the work of Aubin [1] and Yau [15], and when $\mu = 1$ by the work of Yau. Hence the iteration exists for each $k \in \mathbb{N}$.

We divide the discussion into three cases, according to the sign of the first Chern class.

Assume first that $c_1 < 0$ and let $\Omega = -c_1$. For each k write $\omega_k = \omega_{\psi_k}$ with $\psi_k = \sum_{l=1}^k \varphi_l$. We have the following system of Monge–Ampère equations: $\omega_{\psi_k}^n = \omega^n \mathrm{e}^{f_\omega + \psi_k + \varphi_k}$, $k \in \mathbb{N}$. One readily sees that an inductive argument using

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[1] Most of the results hold also for discretizations corresponding to other time intervals.

the maximum principle implies that $\|\psi_k - \psi_{k-1}\|_{C^0} \leqslant C2^{-k}$. This uniform bound implies higher order a priori estimates, by elliptic regularity theory. Therefore the sequence converges exponentially fast to a smooth function that we denote by ψ_∞.

Consider the Chen–Tian functionals E_k [4]. We now observe the following monotonicity result. Its proof can be deduced from some of our previous results [11].

Lemma 2.3. *Along the iteration E_0 is monotonically decreasing whenever $\omega_0 \in \mathcal{H}_\Omega$. When $\mu = 1$ the same is true for E_1, and if $\omega_0 \in \mathcal{H}_{c_1}^+$, also for E_k, $k \geqslant 2$.*

Coming back to the proof of the theorem, note that unless ω_0 is itself Kähler–Einstein, the functional E_0 is strictly decreasing along the iteration. In particular since ω_∞ is a fixed point of the iteration it must be Kähler–Einstein. The case $\mu = 0$ is similar and so we omit the details.

Finally, we turn to the case $\mu = 1$ and assume for simplicity that there are no holomorphic vector fields. In the case $\mu = 1$ the corresponding iteration takes a very special form.

Definition 2.4. Define the inverse Ricci operator $\mathrm{Ric}^{(-1)} : \mathcal{D}_{c_1} \to \mathcal{H}_{c_1}$ by letting $\mathrm{Ric}^{(-1)} \omega := \omega_\varphi$ with ω_φ the unique Kähler form in $\mathcal{H}_{c_1}$ (given by the Calabi–Yau Theorem [15]) satisfying $\mathrm{Ric}\,\omega_\varphi = \omega$. Similarly denote higher order iterates of this operator by $\mathrm{Ric}^{(-l)}$, $l \in \mathbb{Z}$, with $\mathrm{Ric}^{(0)} := \mathrm{Id}$.

We then see that the dynamical system for $\mu = 1$ is nothing but the evolution of iterates of the inverse Ricci operator, $\omega_k = \mathrm{Ric}^{(-k)} \omega_0$. For this case we are solving the system of equations $\omega_{\psi_k}^n = \omega^n e^{f_\omega - \psi_{k-1}}$, $k \in \mathbb{N}$. Let G_k be a Green function for $-\Delta_k := -\Delta_{\omega_{\varphi_k}}$ satisfying $\int_M G_k(\cdot, y)\omega_{\psi_k}^n(y) = 0$. Set $A_k = -\inf_{M \times M} G_k$. Let $I(\omega, \omega_\varphi) = V^{-1} \int_M \varphi(\omega^n - \omega_\varphi^n)$. Application of the Green formula gives $|\psi_k| \leqslant n(A_0 + A_k) + I(\omega_0, \omega_{\psi_k})$. Since E_0 is proper on $\mathcal{H}_{c_1}$ in the sense of Tian [13], we conclude that $I(\omega, \omega_{\psi_k})$ is uniformly bounded. The crucial technical ingredient is now a uniform upper bound on the diameter. Its derivation hinges on properties of the energy functionals, the definition of the iteration, and finally on an argument due to Perelman adapted to this "discrete" situation. Now, combining the diameter estimate with the Green function estimate of Bando and Mabuchi [2], we conclude that $|\psi_k| \leqslant C$. By monotonicity of the energy functionals we conclude that a subsequence can be chosen, converging to a Kähler–Einstein metric. This completes the outline of the proof of Theorem 2.2.

There are a number of applications of these constructions to several well-known objects of study in Kähler geometry, among them canonical metrics, energy functionals, the Moser–Trudinger–Onofri inequality, balanced metrics and the structure of the space of Kähler metrics. We point out two of the most obvious ones and describe others elsewhere. Also, these constructions can be generalized in some interesting directions [12].

The first application is an answer to a question raised by Nadel [8]: Given $\omega \in \mathcal{H}_{c_1}$ define a sequence of metrics $\omega, \mathrm{Ric}\,\omega, \mathrm{Ric}(\mathrm{Ric}\,\omega), \ldots$, as long as positivity is preserved; what are the periodic orbits of this dynamical system? The cases $k = 2, 3$ in the following theorem are due to Nadel:

Theorem 2.5. *Let (M, J, ω) be a Fano manifold and assume that $\mathrm{Ric}^{(k)}(\omega) = \omega$ for some $k \in \mathbb{Z}$. Then ω is Kähler–Einstein.*

This is an immediate corollary of Lemma 2.3.

The second application is the construction of multiplier ideal sheaves [7] when a Kähler–Einstein metric does not exist. It may be seen as a discrete counterpart to a recent result of Phong–Šešum–Sturm [9].

Theorem 2.6. *Let (M, J) be a Fano manifold that does not admit a Kähler–Einstein metric and let $\gamma > 1$. One may extract a subsequence $\{\psi_{k_j}\}$ such that $\lim_{j \to \infty} \psi_{k_j} = \psi_\infty$ exists in $L^1_{\mathrm{loc}}(M)$ and defines a nontrivial Nadel-type multiplier ideal sheaf defined for each open set $U \subseteq M$ by local sections $\{h \in \mathcal{O}_M(U) : |h|^2 e^{-\gamma \psi_\infty} \in L^1_{\mathrm{loc}}(M)\}$.*

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References

[1] T. Aubin, Équations du type Monge–Ampère sur les variétés kählériennes compactes, Bull. Sci. Math. 102 (1978) 63–95.

[2] S. Bando, T. Mabuchi, Uniqueness of Kähler–Einstein metrics modulo connected group actions, in: Algebraic Geometry, Sendai, 1985, Kinokuniya, 1987, pp. 11–40.

[3] H.-D. Cao, Deformations of Kähler metrics to Kähler–Einstein metrics on compact Kähler manifolds, Invent. Math. 81 (1985) 359–372.

[4] X.-X. Chen, G. Tian, Ricci flow on Kähler–Einstein surfaces, Invent. Math. 147 (2002) 487–544.

[5] D.Z.-D. Guan, Extremal-solitons and C^∞ convergence of the modified Calabi flow on certain CP^1 bundles, preprint, December 22, 2006.

[6] R.S. Hamilton, Three-manifolds with positive Ricci curvature, J. Diff. Geom. 17 (1982) 255–306.

[7] A.M. Nadel, Multiplier ideal sheaves and Kähler–Einstein metrics of positive scalar curvature, Ann. Math. 132 (1990) 549–596.

[8] A.M. Nadel, On the absence of periodic points for the Ricci curvature operator acting on the space of Kähler metrics, in: Modern Methods in Complex Analysis, Princeton University Press, 1995, pp. 277–281.

[9] D.H. Phong, N. Šešum, J. Sturm, Multiplier ideal sheaves and the Kähler–Ricci flow, preprint, arxiv: math.DG/0611794.

[10] Y.A. Rubinstein, On iteration of the Ricci operator on the space of Kähler metrics, I, manuscript, August 14, 2005, unpublished.

[11] Y.A. Rubinstein, On energy functionals, Kähler–Einstein metrics and the Moser–Trudinger–Onofri neighborhood, preprint, arxiv: math.DG/0612440.

[12] Y.A. Rubinstein, Some discretizations of geometric evolution equations and the Ricci iteration on the space of Kähler metrics I, preprint, 2007; II, preprint, in preparation.

[13] G. Tian, Kähler–Einstein metrics with positive scalar curvature, Invent. Math. 130 (1997) 1–37.

[14] G. Tian, X.-H. Zhu, Convergence of Kähler–Ricci flow, J. Amer. Math. Soc. 20 (2007) 675–699.

[15] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampére equation, I, Comm. Pure Appl. Math. 31 (1978) 339–411.