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Partial Differential Equations

Analytic continuation of the resolvent of the Laplacian and the dynamical zeta function

Vesselin Petkov [a], Latchezar Stoyanov [b]

[a] *Université Bordeaux I, Institut de mathématiques, 351, cours de la Libération, 33405 Talence, France*
[b] *University of Western Australia, School of Mathematics and Statistics, Perth, WA 6009, Australia*

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Abstract

Let $s_0 < 0$ be the abscissa of absolute convergence of the dynamical zeta function $Z(s)$ for several disjoint strictly convex compact obstacles $K_i \subset \mathbb{R}^N,\ i = 1, \dots, \kappa_0,\ \kappa_0 \geqslant 3$ and let $R_\chi(z) = \chi(-\Delta_D - z^2)^{-1}\chi,\ \chi \in C_0^\infty(\mathbb{R}^N)$, be the cut-off resolvent of the Dirichlet Laplacian $-\Delta_D$ in $\Omega = \overline{\mathbb{R}^N \setminus \bigcup_{i=1}^{\kappa_0} K_i}$. We prove that there exists $\sigma_2 < s_0$ such that $Z(s)$ is analytic for $\Re(s) \geqslant \sigma_2$ and the cut-off resolvent $R_\chi(z)$ has an analytic continuation for $\Im(z) < -\mathbf{i}\sigma_2,\ |\Re(z)| \geqslant C$. ***To cite this article: V. Petkov, L. Stoyanov, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Prolongement analytique de la résolvante du Laplacien et de la fonction zeta dynamique. Soit $s_0 < 0$ l'abscisse de convergence absolue de la fonciton zeta dynamique $Z(s)$ pour des obstacles compacts, disjoints et strictement convexes $K_i \subset \mathbb{R}^N,\ i = 1, \dots, \kappa_0,\ \kappa_0 \geqslant 3$ et soit $R_\chi(z) = \chi(-\Delta_D - z^2)^{-1}\chi,\ \chi \in C_0^\infty(\mathbb{R}^N)$, la résolvante tronquée du Laplacien de Dirichlet $-\Delta_D$ dans $\Omega = \overline{\mathbb{R}^N \setminus \bigcup_{i=1}^{\kappa_0} K_i}$. On prouve qu'il existe $\sigma_2 < s_0$ tel que $Z(s)$ est analytique pour $\Re(s) \geqslant \sigma_2$ et la résolvante tronquée $R_\chi(z)$ admet un prolongement analytique pour $\Im(z) < -\mathbf{i}\sigma_2,\ |\Re(z)| \geqslant C$. ***Pour citer cet article : V. Petkov, L. Stoyanov, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

Soit $K = K_1 \cup K_2 \cup \cdots \cup K_{k_0}$, où $K_i \subset \mathbb{R}^N,\ N \geqslant 2$, sont des domaines compacts, disjoints et strictement convexes ayant des frontières $\Gamma_i = \partial K_i$ et $\kappa_0 \geqslant 3$. Soit $\Omega = \overline{\mathbb{R}^N \setminus K}$ et $\Gamma = \partial K$. On suppose que K satisfait la condition suivante :

(H) Pour chaque couple K_i, K_j de différentes composantes connexes de K l'enveloppe convexe de $K_i \cup K_j$ n'a pas de points communs avec les autres composantes connexes de K.

E-mail addresses: petkov@math.u-bordeaux1.fr (V. Petkov), stoyanov@maths.uwa.edu.au (L. Stoyanov).

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Étant donné un rayon périodique réfléchissant $\gamma \subset \Omega$ avec m_γ réflexions, soit T_γ la période primitive (longueur) de γ et soit P_γ l'application de Poincaré linéaire associée à γ (cf. [8]). Notons par $\lambda_{i,\gamma},\ i = 1,\dots,N-1$, les valeurs propres de P_γ telles que $|\lambda_{i,\gamma}| > 1$ et désignons par $\mathcal{P}$ l'ensemble de rayons primitifs périodiques. Soit $\delta_\gamma = -\frac{1}{2}\log(\lambda_{1,\gamma}\cdots\lambda_{N-1,\gamma})$, $r_\gamma = 0$ si m_γ est pair et $r_\gamma = 1$ si m_γ est impair. On considère la fonction zeta dynamique

$$Z(s) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\gamma\in\mathcal{P}} (-1)^{mr_\gamma} \mathrm{e}^{m(-sT_\gamma+\delta_\gamma)}.$$

On voit facilement qu'il existe une abscisse de convergence absolue $s_0 \in \mathbb{R}$ telle que pour $\Re(s) > s_0$ la série $Z(s)$ est absolument convergente. D'autre part, en utilisant la dymanique symbolique et les résultats de [7], on conclut que $Z(s)$ est méromorphe pour $\Re(s) > s_0 - a,\ a > 0$ (cf. [4]). En suivant les résultats récents (cf. [9] pour $N = 2$ et [10] pour $N = 3$ sous certaines conditions) on sait qu'il existe $0 < \epsilon < a$ tel que la fonction zeta dynamique $Z(s)$ admet un prolongement analytique pour $\Re(s) > s_0 - \epsilon$. On considère maintenant pour $\Im(s) < 0$ la résolvante tronquée $R_\chi(z) = \chi(-\Delta_D - z^2)^{-1}\chi : L^2(\Omega) \to L^2(\Omega)$, où $\chi \in C_0^\infty(\mathbb{R}^N),\ \chi = 1$ sur K et $-\Delta_D$ est le Laplacien de Dirirchlet dans $\Omega = \overline{\mathbb{R}^N \setminus K}$. La résolvante $R_\chi(z)$ admet un prolongement mémororphe dans $\mathbb{C}$ pour N impair et dans $\mathbb{C}\setminus\{\mathrm{i}\mathbb{R}^+\}$ pour N pair avec des pôles $z_j,\ \Im(z_j) > 0$. On se propose d'étudier la liaison entre les prolongements analytiques de $Z(s)$ et $R_\chi(z)$. Le cas $s_0 > 0$ est plus facile car on sait que pour $-\mathbf{i}s_0 \leqslant \Im(z) < 0$ la résolvante tronquée $R_\chi(z)$ est analytique [6]. Dans la suite on suppose que $s_0 < 0$. Sous l'hypothèse $s_0 < 0$, Ikawa [3] a démontré que pour tout $\epsilon > 0$ il existe $C_\epsilon > 0$ tel que $R_\chi(z)$ est analytique pour $\Im(z) < -\mathbf{i}(s_0 + \epsilon)$, $|\Re(z)| \geqslant C_\epsilon$. Un résultat similaire pour un problème du contrôle a été établi par Burq [1]. La fonction zeta dynamique $Z(s)$ est liée aux périodes des rayons périodiques et formellement $Z(s)$ ne contient pas une information sur la dynamique de rayons dans un voisinage de l'ensemble 'non-wandering' (captif). Dans cette Note on examine le cas $\Re(s) < s_0$ en exploitant les propriétés spectrales de l'opérateur de Ruelle L_s (cf. Section 2). Notre résultat principal est le suivant :

Théorème 0.1. *Soit $s_0 < 0$. Supposons que l'opérateur de Ruelle L_s satisfait les estimations* (6). *Alors il existe $\sigma_2 < s_0$ tel que $Z(s)$ est analytique pour $\Re(s) > \sigma_2$ et la résolvante tronquée $R_\chi(z)$ est analytique pour*

$$\Im(z) < -\mathbf{i}\sigma_2, \qquad \left|\Re(s)\right| \geqslant C.$$

Les estimations (6) sont un analogue aux estimations de Dolgopyat [2]. Ces estimaitons ont été démontrées pour $N = 2$ dans [9] et pour $N \geqslant 3$ sous certaines conditions dans [10]. On espère que (6) sont valables pour $N \geqslant 3$ sans aucune restriction. Notons qu'il y a quelques ans, Ikawa [5] a annoncé un résultat concernant le prolongement analytique de $R_\chi(z)$ dans un domaine $-\mathbf{i}\mathcal{D}_{\epsilon,\alpha}$, où

$$\mathcal{D}_{\epsilon,\alpha} = \left\{ s \in \mathbb{C}\colon\ \Re(s) > s_0 - \left|\Im(s)\right|^{-\alpha},\ \left|\Im(s)\right| \geqslant C_\epsilon,\ 0 < \alpha < 1 \right\}$$

en imposant des conditions fortes sur le comportement de la fonction propre w de l'opérateur de Ruelle associée à la valeur propre maximale et un prolongement analytique de $Z(s)$ dans $\mathcal{D}_{\epsilon,\alpha}$. De plus, il suppose qu'on ait l'estimation $|Z(s)| \leqslant |s|^{1-\epsilon},\ 0 < \epsilon < 1,\ s \in \mathcal{D}_{\epsilon,\alpha}$. A notre connaissance la preuve de ce résultat n'a pas été publiée ailleurs.

1. Introduction

Let K be a subset of $\mathbb{R}^N,\ N \geqslant 2$, of the form $K = K_1 \cup K_2 \cup \cdots \cup K_{k_0}$, where K_i are compact strictly convex disjoint domains in $\mathbb{R}^N$ with C^∞ boundaries $\Gamma_i = \partial K_i$ and $k_0 \geqslant 3$. Set $\Omega = \overline{\mathbb{R}^N \setminus K}$ and $\Gamma = \partial K$. We assume that K satisfies the following (no-eclipse) condition:

(H) For every pair K_i, K_j of different connected components of K the convex hull of $K_i \cup K_j$ has no common points with any other connected component of K.

With this condition, the billiard flow ϕ_t defined on the cosphere bundle $S^*(\Omega)$ in the standard way is called an open billiard flow. Given a periodic reflecting ray $\gamma \subset \Omega$ with m_γ reflections, denote by T_γ the primitive period (length) of γ and by P_γ the linear Poincaré map associated to γ (see [8]). Let $\lambda_{i,\gamma}, i = 1,\dots,N-1$, be the eigenvalues of P_γ

with $|\lambda_{i,\gamma}| > 1$ and let $\mathcal{P}$ be the set of primitive periodic rays. For $\gamma \in \mathcal{P}$ set $\delta_\gamma = -\frac{1}{2}\log(\lambda_{1,\gamma}\cdots\lambda_{N-1,\gamma})$, $r_\gamma = 0$ if m_γ is even and $r_\gamma = 1$ if m_γ is odd. Consider the dynamical zeta function

$$Z(s) = \sum_{m=1}^{\infty}\frac{1}{m}\sum_{\gamma\in\mathcal{P}}(-1)^{mr_\gamma}\mathrm{e}^{m(-sT_\gamma+\delta_\gamma)}.$$

It is easy to show that there exists an abscissa of absolute convergence $s_0 \in \mathbb{R}$ such that for $\Re(s) > s_0$ the series $Z(s)$ is absolutely convergent. On the other hand, using symbolic dynamics and the results of [7], we deduce that $Z(s)$ is meromorphic for $\Re(s) > s_0 - a,\ a > 0$ (see [4]). According to some recent results (see [9] for $N = 2$ and [10] for $N \geqslant 3$ under some additional conditions) there exists $0 < \epsilon < a$ so that the dynamical zeta function $Z(s)$ admits an analytic continuation for $\Re(s) \geqslant s_0 - \epsilon$. For $\Im(z) < 0$ consider the cut-off resolvent $R_\chi(z) = \chi(-\Delta_D - z^2)^{-1}\chi : L^2(\Omega) \to L^2(\Omega)$, where $\chi \in C_0^\infty(\mathbb{R}^N),\ \chi = 1$ on K and $-\Delta_D$ is the Dirichlet Laplacian in $\Omega = \overline{\mathbb{R}^N \setminus K}$. The cut-off resolvent $R_\chi(z)$ has a meromorphic continuation in $\mathbb{C}$ for N odd and in $\mathbb{C} \setminus \{\mathrm{i}\mathbb{R}^+\}$ for N even with poles z_j such that $\Im(z_j) > 0$. The analytic properties and the estimates of $R_\chi(z)$ play a crucial role in many problems related to the local energy decay, distribution of the resonances etc. We study the link between the analytic continuations of $Z(s)$ and $R_\chi(z)$. The case $s_0 > 0$ is much easier, since we know that for $-\mathbf{i}s_0 \leqslant \Im(z) < 0$ the cut-off resolvent $R_\chi(z)$ is analytic (see [6]).

In the following **we assume that** $s_0 < 0$. The problem is to examine the link between the analyticity of $Z(s)$ for $\Re(s) > s_0$ and the behavior of $R_\chi(z)$ for $0 \leqslant \Im(z) < -\mathbf{i}s_0$. Assuming $s_0 < 0$, Ikawa [3] proved that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ so that the cut-off resolvent $R_\chi(z)$ is analytic for $\Im(z) < -\mathbf{i}(s_0 + \epsilon)$, $|\Re(z)| \geqslant C_\epsilon$. A similar result for a control problem has been established by Burq [1]. The proofs in [3] and [1] are based on the construction of an asymptotic solution $U_M(x, s; k)$ with boundary data $m(x; k) = \mathrm{e}^{\mathbf{i}k\varphi(x)}h(x),\ k \in \mathbb{R},\ k \geqslant 1$, where φ is a phase function ($\|\nabla\varphi\| = 1$) and $h \in C^\infty(\Gamma)$ has a small support. More precisely, $U_M(\cdot, s; k)$ is $C^\infty(\bar{\Omega})$-valued holomorphic function in $\mathcal{D}_0 = \{s \in \mathbb{C}\colon\ \Re(s) > s_0\}$, and we have

$$(\Delta - s^2)U_M(\cdot,s;k) = 0 \quad \text{for } x \in \Omega,\ \Re(s) > s_0, \tag{1}$$

$$U_M(\cdot,s;k) \in L^2(\Omega) \quad \text{if } \Re(s) > 0, \tag{2}$$

$$U_M(x,s;k) = m(x,k) + r_M(x,s;k) \quad \text{on } \Gamma, \tag{3}$$

where, for $r_M(x, s; k)$ and $s \in \mathcal{D}_0,\ |s + \mathbf{i}k| \leqslant 1$, we have the estimates

$$\big\|r_M(\cdot,s;k)\big\|_{C^p(\Gamma)} \leqslant C_p k^{-M+p}\big(\|\nabla\varphi\|_{C^{M^2+p+2}(\Gamma)} + 1\big)\|h\|_{C^{M^2+p+2}(\Gamma)}, \quad \forall p \in \mathbb{N}. \tag{4}$$

The function $U_M(x, s; k)$ is given by a finite sum of series having the form

$$\sum_{n=0}^{\infty}\sum_{|\mathbf{j}|=n+2,\,\mathbf{j}_{n+2}=l}\sum_{q=0}^{M}\mathrm{e}^{-s\varphi_{\mathbf{j}}(x)}\sum_{\nu=0}^{2q}\big(a_{\mathbf{j},q,\nu}(x,s;k)(s+\mathbf{i}k)^\nu\big)(\mathbf{i}k)^{-q}, \tag{5}$$

where $\mathbf{j} = (j_1, \ldots, j_m),\ j_i \in \{1, \ldots, \kappa_0\}$ are configurations, $|\mathbf{j}| = m,\ \varphi_{\mathbf{j}}(x)$ are phase functions and the amplitudes $a_{\mathbf{j},q,\nu}(x, s; k)$ depend on $s \in \mathbb{C}$ and $k \in \mathbb{R}$. The main difficulty is to establish the summability of these series and to obtain for $\Re(s) > s_0$ suitable C^p estimates of their traces on Γ. The absolute convergence of $Z(s)$ makes it possible to establish the absolute convergence of the series in (5) and to get crude estimates leading to (1)–(4). The dynamical zeta function $Z(s)$ is related to the periods of periodic rays and formally from $Z(s)$ we get no information about the dynamics of all rays in a *neighborhood of the non-wandering* (*trapped*) *set*. In this Note we study the case $\Re(s) < s_0$ by means of the Ruelle operator L_s (see Section 2 for the definition). Our main result is the following:

Theorem 1.1. *Let* $s_0 < 0$. *Assume that for the Ruelle operator* L_s *the estimates* (6) *hold. Then there exists* $\sigma_2 < s_0$ *such that* $Z(s)$ *is analytic for* $\Re(s) > \sigma_2$ *and the cut-off resolvent* $R_\chi(z)$ *is analytic for*

$$\Im(z) < -\mathbf{i}\sigma_2, \qquad \big|\Re(s)\big| \geqslant C.$$

The estimates (6) are analogous to Dolgopyat's estimates in [2]. For open billiard flows (6) have been established in [9] for $N = 2$ and under some conditions in [10] for $N \geqslant 3$. We expect that (6) hold for $N \geqslant 3$ without any restrictions.

Several years ago, Ikawa [5] announced a result concerning an analytic continuation of $R_\chi(z)$ in a domain $-\mathbf{i}\mathcal{D}_{\epsilon,\alpha}$, where

$$\mathcal{D}_{\epsilon,\alpha} = \{s \in \mathbb{C}\colon \Re(s) > s_0 - |\Im(s)|^{-\alpha}, |\Im(z)| \geqslant C_\epsilon,\ 0 < \alpha < 1\}$$

assuming some strong conditions on the behavior of the eigenfunction w of the corresponding Ruelle operator related to its maximal eigenvalue as well as an analytic continuation of $Z(s)$ in $\mathcal{D}_{\epsilon,\alpha}$ combined with an estimate $|Z(s)| \leqslant |s|^{1-\epsilon},\ 0 < \epsilon < 1,\ s \in \mathcal{D}_{\epsilon,\alpha}$. To our best knowledge a proof of the above result of Ikawa has not been published anywhere.

2. Ruelle operator

Introduce the spaces

$$\Sigma_A = \{(\dots, \eta_{-m}, \dots, \eta_{-1}, \eta_0, \eta_1, \dots, \eta_m, \dots)\colon 1 \leqslant \eta_j \leqslant \kappa_0,\ \eta_j \in \mathbb{N},\ \eta_j \neq \eta_{j+1} \text{ for all } j \in \mathbb{Z}\},$$
$$\Sigma_A^+ = \{(\eta_0, \eta_1, \dots, \eta_m, \dots)\colon 1 \leqslant \eta_j \leqslant \kappa_0,\ \eta_j \in \mathbb{N},\ \eta_j \neq \eta_{j+1} \text{ for all } j \geqslant 1\}.$$

We define the operator $\sigma : \Sigma_A^+ \to \Sigma_A^+$ by $(\sigma\xi)_i = \xi_{i+1},\ i \in \mathbb{N}$. Given $\xi \in \Sigma_A$, let

$$\dots, P_{-2}(\xi), P_{-1}(\xi), P_0(\xi), P_1(\xi), P_2(\xi), \dots$$

be the successive reflection points of the unique billiard trajectory in the exterior of K such that $P_j(\xi) \in K_{\xi_j}$ for all $j \in \mathbb{Z}$. Set $f(\xi) = \|P_0(\xi) - P_1(\xi)\|$. Following [4], one constructs a sequence $\{\varphi_{\xi,j}\}_{j=-\infty}^{\infty}$ of phase functions such that for each j, $\varphi_{\xi,j}$ is defined and smooth in a neighborhood $U_{\xi,j}$ of the segment $[P_j(\xi), P_{j+1}(\xi)]$ in Ω and

(i) $\nabla\varphi_{\xi,j}(P_j(\xi)) = \frac{P_{j+1}(\xi)-P_j(\xi)}{\|P_{j+1}(\xi)-P_j(\xi)\|}$,
(ii) $\varphi_{\xi,j} = \varphi_{\xi,j+1}$ on $\Gamma_{\xi_{j+1}} \cap U_{\xi,j} \cap U_{\xi,j+1}$,
(iii) for each $x \in U_{\xi,j}$ the surface $C_{\xi,j}(x) = \{y \in U_{\xi,j}\colon \varphi_{\xi,j}(y) = \varphi_{\xi,j}(x)\}$ is strictly convex with respect to its normal field $\nabla\varphi_{\xi,j}$. For any $y \in U_{\xi,j}$ denote by $G_{\xi,j}(y)$ the *Gauss curvature* of $C_{\xi,j}(x)$ at y.

Now define $g : \Sigma_A \to \mathbb{R}$ by

$$g(\xi) = \frac{1}{N-1} \ln \frac{G_{\xi,0}(P_1(\xi))}{G_{\xi,0}(P_0(\xi))}.$$

By Sinai's Lemma, there exist $\tilde{f}, \tilde{g}$ depending on future coordinates only and χ_1, χ_2 such that

$$f(\xi) = \tilde{f}(\xi) + \chi_1(\xi) - \chi_1(\sigma\xi), \quad g(\xi) = \tilde{g}(\xi) + \chi_2(\xi) - \chi_2(\sigma\xi), \quad \xi \in \Sigma_A.$$

Setting $\tilde{r}(\xi, s) = -s\tilde{f}(\xi) + \tilde{g}(\xi) + \mathbf{i}\pi$, we define the *Ruelle transfer operator* $L_s : C(\Sigma_A^+) \to C(\Sigma_A^+)$ by $L_s u(\xi) = \sum_{\sigma\eta=\xi} \mathrm{e}^{\tilde{r}(\eta,s)} u(\eta)$ for any continuous (complex-valued) function u on Σ_A^+ and any $\xi \in \Sigma_A^+$. For our analysis the Dolgopyat type estimates [2] for the norms of L_s^n play a crucial role. Following the results in [9,10] there exist constants $C > 0, \sigma_0 < s_0$ and $0 < \rho < 1$ so that for $s = \tau + \mathbf{i}t$ with $\tau \geqslant \sigma_0$ and $n = p[\log|t|] + l,\ p \in \mathbb{N},\ 0 \leqslant l \leqslant [\log|t|] - 1$, we have

$$\|L_s^n\|_\infty \leqslant C\rho^{p[\log|t|]} \mathrm{e}^{l \Pr(-\tau\tilde{f}+\tilde{g})}, \quad |t| \geqslant t_0, \tag{6}$$

$\Pr(G)$ being the topological pressure of the function G (see [7]).

3. Idea of the proof of Theorem 1.1

Fix $l \in \{1, \dots, \kappa_0\}$. Given a phase function $\varphi(x)$ and an amplitude $h(x) \in C^\infty(\Gamma)$, we wish to construct an asymptotic solution $U_M(x, s; k)$ which is a holomorphic function for $\Re(s) \geqslant \sigma_2$ and U_M has properties similar to (1)–(4). The first approximation of U_M is an infinite sum

$$w_{0,l}(x, -\mathbf{i}s) = \sum_{n=0}^{\infty} \sum_{|\mathbf{j}|=n+2,\ \mathbf{j}_{n+2}=l} u_{\mathbf{j}}(x, -\mathbf{i}s),$$

where $u_{\mathbf{j}}(x,-\mathbf{i}s)=(-1)^{m-1}\mathrm{e}^{-s\varphi_{\mathbf{j}}(x)}a_{\mathbf{j}}(x)$ is related to a configuration $\mathbf{j}=\{j_1,\dots,j_m\}$ by using successive phase functions $\varphi_{\mathbf{j}}(x)$ and amplitudes $a_{\mathbf{j}}(x)$ determined by the transport equation (see [5]). To justify the convergence of this series, we need to compare the general term with a suitable composition of operators related to the dynamics. Let $\mu=(\mu_0=1,\mu_1,\dots)\in\Sigma_A^+$. It follows from [3] that there exists a unique point $y(\mu)\in\Gamma_1$ such that the ray $\gamma(y,\varphi)$ issued from a point $y(\mu)$ in direction $\nabla\varphi(y(\mu))$ follows the configuration μ. Let $Q_0(\mu)=y(\mu), Q_1(\mu),\dots,$ be the consecutive reflection points of this ray. Define $f_j^+(\mu)=\|Q_j(\mu)-Q_{j+1}(\mu)\|$, and

$$g_j^+(\mu)=\frac{1}{N-1}\ln\frac{G^{\varphi}_{\mu,j}(Q_{j+1}(\mu))}{G^{\varphi}_{\mu,j}(Q_j(\mu))}<0,$$

where $G^{\varphi}_{\mu,j}(y)$ denotes the *Gauss curvature* of the surface $C^{\varphi}_{\mu,j}(x)=\{z\colon\ \varphi_{(\mu_0,\mu_1,\dots,\mu_j)}(z)=\varphi_{(\mu_0,\mu_1,\dots,\mu_j)}(x)\}$ at y. We define an extension $e:\Sigma_A^+\to\Sigma_A$. For $s\in\mathbb{C}$ and $\xi\in\Sigma_A^+$ with $\xi_0=1$, following [5], set

$$\phi^+(\xi,s)=\sum_{n=0}^{\infty}\bigl(-s\bigl[f\bigl(\sigma^n e(\xi)\bigr)-f_n^+(\xi)\bigr]+\bigl[g\bigl(\sigma^n e(\xi)\bigr)-g_n^+(\xi)\bigr]\bigr).$$

Formally, we define $\phi^+(\xi,s)=0$ when $\xi_0\neq 1$, thus obtaining a function $\phi^+:\Sigma_A^+\times\mathbb{C}\to\mathbb{C}$. Set $\chi(\xi,s)=-s\chi_1(\xi)+\chi_2(\xi)$ and for any $s\in\mathbb{C}$ define the operator $\mathcal{G}_s:C(\Sigma_A^+)\to C(\Sigma_A^+)$ by

$$\mathcal{G}_s v(\xi)=\sum_{\sigma\eta=\xi,\,\eta_0=1}\mathrm{e}^{-\phi^+(\eta,s)+\chi(e(\eta),s)-s\tilde f(\eta)+\tilde g(\eta)}v(\eta),\quad v\in C(\Sigma_A^+),\ \xi\in\Sigma_A^+.$$

Fix an arbitrary $l\in\{1,\dots,\kappa_0\}$ **and an arbitrary point** $x_0\in\Gamma_l$. Define the function $\phi^-(x_0;\cdot,\cdot):\Sigma_A\times\mathbb{C}\to\mathbb{C}$ (depending on l as well) as follows. First, set $\phi^-(x_0;\eta,s)=0$ if $\eta_0\neq l$. Next, assume that $\eta\in\Sigma_A$ satisfies $\eta_0=l$. There exists a unique billiard trajectory in Ω with successive reflection points $\tilde P_j(x_0;\eta)\in\partial K_{\eta_j}$ $(-\infty<j\leqslant 0)$ such that $x_0=\tilde P_{-1}(x_0;\eta)+t\nabla\varphi_{\eta^-}(\tilde P_{-1}(x_0;\eta))$ for some $t>0$. In general the segment $[\tilde P_{-1}(x_0;\eta),x_0]$ may intersect the interior of K_l. If this is the case, set again $\phi^-(x_0;\eta,s)=0$. Otherwise, denote $\tilde P_0(x_0;\eta)=x_0$ and for any $j<0$ set

$$f_j^-(x_0;\eta)=\bigl\|\tilde P_{j+1}(x_0;\eta)-\tilde P_j(x_0;\eta)\bigr\|,\qquad g_j^-(x_0;\eta)=\frac{1}{N-1}\ln\frac{G_{\eta,j}(\tilde P_{j+1}(x_0;\eta))}{G_{\eta,j}(\tilde P_j(x_0;\eta))},$$

and define $\phi^-(x_0;\eta,s)=-s\sum_{j=-1}^{-\infty}[f(\sigma^j(\eta))-f_j^-(x_0;\eta)]+\sum_{j=-1}^{-\infty}[g(\sigma^j(\eta))-g_j^-(x_0;\eta)]$.

Next, similarly to [5], introduce the operator $\mathcal{M}_{n,s}(x_0):C(\Sigma_A^+)\to C(\Sigma_A^+)$ by

$$\bigl(\mathcal{M}_{n,s}(x_0)v\bigr)(\xi)=\sum_{\sigma\eta=\xi}\mathrm{e}^{-\phi^-(x_0;\sigma^{n+1}e(\eta),s)-\chi(\sigma^{n+1}e(\eta),s)-s\tilde f(\eta)+\tilde g(\eta)}v(\eta),\quad v\in C(\Sigma_A^+),\ \xi\in\Sigma_A^+.$$

Introduce the function $v_s(\xi)=\mathrm{e}^{-s\varphi(Q_0(\xi))}h(Q_0(\xi))$ if $\xi_0=1$ and $v_s(\xi)=0$ otherwise and define the norms

$$\|f\|_{\Gamma,p}=\max_{x\in\Gamma}\max_{a^{(1)},\dots,a^{(p)}\in T_x\Gamma}\bigl\|(D_{a^{(1)}}\cdots D_{a^{(p)}}f)(x)\bigr\|,\qquad \|f\|_{\Gamma,(p)}=\max_{0\leqslant j\leqslant p}\|f\|_{\Gamma,j}$$

where $\|a^{(j)}\|=1$ for all $j=1,\dots,p$.

Theorem 3.1. *There exist global constants $C>0$, $c>0$, $a\in(0,1]$ and $\theta\in(0,1)$ depending only on K such that for any choice of $l\in\{1,\dots,\kappa_0\}$ the following holds: For any integers $p\geqslant 1$ and $n\geqslant 1$, any $\xi\in\Sigma_A^+$ with $\xi_0=l$ and any $s\in\mathbb{C}$ with $\Re(s)\geqslant s_0-a$ we have*

$$\Bigl|\bigl(L_s^n\mathcal{M}_{n,s}(\cdot)\mathcal{G}_s\tilde v_s\bigr)(\xi)-\sum_{|\mathbf{j}|=n+2,\,\mathbf{j}_{n+2}=l}u_{\mathbf{j}}(\cdot,-\mathbf{i}s)\Bigr|_{\Gamma,p}$$
$$\leqslant C(\theta+ca)^n\mathrm{e}^{C[|\Re(s)|(1+\|\varphi\|_{\Gamma,0})+\|\nabla\varphi\|_{\Gamma,(1)}]}\sum_{j=0}^{p}\bigl(|s|\|\nabla\varphi\|_{\Gamma,j}+\|\nabla\varphi\|_{\Gamma,j+1}\bigr)^{j+1}\|h\|_{\Gamma,p-j}.\tag{7}$$

A similar estimate holds for $p = 0$; in this case the sum in the right-hand side of (7) has to be replaced by $[(|s| + \|\nabla\varphi\|_{\Gamma,(1)})\|h\|_{\Gamma,0} + \|h\|_{\Gamma,(1)}]$. Applying the case $p = 0$, we reduce the convergence of $w_{0,l}$ to the summability of the series $\sum_{n=0}^{\infty} L_s^n$. On the other hand, for $\tau \geqslant \sigma_0$, $|t| \geqslant 2$ the estimates (6) yield

$$\sum_{n=0}^{\infty} \|L_s^n\|_\infty \leqslant \frac{C}{1-\rho^{[\log|t|]}} \sum_{j=0}^{[\log|t|]-1} \mathrm{e}^{j\,\mathrm{Pr}(-\tau\tilde{f}+\tilde{g})} \leqslant C_1 \max\bigl\{\log|t|, |t|^{\mathrm{Pr}(-\tau\tilde{f}+\tilde{g})}\bigr\}.$$

Moreover, for σ_0 sufficiently close to s_0 there exists $0 < \beta < 1$ such that $\|L_s^n \mathcal{M}_{n,s}\mathcal{G}\|_{\Gamma,0} \leqslant C|t|^{1+\beta}$ and we conclude that $\|w_{0,l}(x, -\mathbf{i}\tau + t)\|_{\Gamma_l,0} \leqslant B|t|^{1+\beta}$. Exploiting the case $p \geqslant 1$, we obtain similar estimates for $\|w_{0,l}(x, -\mathbf{i}\tau + t)\|_{\Gamma_l,p}$, $p \geqslant 1$ and we get the first approximation. Repeating this procedure, we complete the construction of U_M.

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