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Differential Geometry

On the continuity of the second Sobolev best constant

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Abstract

In this Note we prove that the second Riemannian L^p-Sobolev best constant $B_0(p, g)$ depends continuously on g in the C^0-topology when $1 < p < 2$. The situation changes significantly in the case $p = 2$. In particular, we prove that $B_0(2, g)$ is continuous on g in the C^2-topology and is not in the $C^{1,\beta}$-topology. ***To cite this article: E.R. Barbosa, M. Montenegro, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Sur la continuité de la deuxième meilleure constante de Sololev. Dans cette Note nous prouvons que la deuxième meilleure constante dans l'inégalité de L^p-Sobolev Riemannienne $B_0(p, g)$ dépend continûment de g dans la topologie C^0 quand $1 < p < 2$. La situation change radicalement lorsque $p = 2$. En particulier, nous montrons que $B_0(2, g)$ est continu en g dans le C^2-topologie et ne l'est pas dans le $C^{1,\beta}$-topologie. ***Pour citer cet article : E.R. Barbosa, M. Montenegro, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction and main results

Let (M, g) be a compact Riemannian manifold of dimension $n \geqslant 2$. For $1 < p < n$, we denote by $H_1^p(M)$ the standard first-order Sobolev space defined as the completion of $C^\infty(M)$ with respect to the norm

$$||u||_{H_1^p(M)} = \left(\int_M |\nabla_g u|^p \, dv_g + \int_M |u|^p \, dv_g\right)^{1/p}.$$

The Sobolev embedding theorem ensures that the inclusion $H_1^p(M) \subset L^{p^*}(M)$ is continuous for $p^* = \frac{np}{n-p}$. Thus, there exist constants $A, B \in \mathbb{R}$ such that, for any $u \in H_1^p(M)$,

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$$\left(\int_M |u|^{p^*}\,\mathrm{d}v_g\right)^{p/p^*} \leqslant A\int_M |\nabla_g u|^p\,\mathrm{d}v_g + B\int_M |u|^p\,\mathrm{d}v_g. \tag{I_g^p}$$

In this case, we say simply that (I_g^p) is valid.

The first Sobolev best constant associated to (I_g^p) is

$$A_0(p,g) = \inf\bigl\{A \in \mathbb{R}\colon \text{ there exists } B \in \mathbb{R} \text{ such that } (I_g^p) \text{ is valid}\bigr\}.$$

The value of $A_0(p,g)$ was found by Aubin in [1]. This best constant is usually denoted in the literature by $K(n,p)^p$ since its value does not depend on the metric g.

The first optimal Riemannian L^p-Sobolev inequality states that, for any $u \in H_1^p(M)$,

$$\left(\int_M |u|^{p^*}\,\mathrm{d}v_g\right)^{p/p^*} \leqslant K(n,p)^p\int_M |\nabla_g u|^p\,\mathrm{d}v_g + B\int_M |u|^p\,\mathrm{d}v_g \tag{$I_{g,\mathrm{opt}}^p$}$$

for some constant $B \in \mathbb{R}$. The validity of $(I_{g,\mathrm{opt}}^p)$ has been proved, for $p = 2$, by Hebey and Vaugon [8], and for $1 < p < 2$, independently, by Aubin and Li [2] and Druet [6]. When $2 < p < \frac{n+2}{3}$ and the scalar curvature of g is positive somewhere, Druet [5] showed the non-validity of $(I_{g,\mathrm{opt}}^p)$.

For $1 < p \leqslant 2$, define the second L^p-Sobolev best constant by $B_0(p,g) = \inf\{B \in \mathbb{R}\colon (I_{g,\mathrm{opt}}^p) \text{ is valid}\}$. On the contrary of the first Sobolev best constant, the second one depends strongly on the metric. Note that if $\tilde{g} = \lambda g$, where $\lambda > 0$ is a constant, then $B_0(p,\tilde{g}) = \lambda^{-1}B_0(p,g)$. Thus, the following question arises naturally: Does $B_0(p,g)$ depend continuously on the metric g in some topology? Surprising, the answer to this question changes significantly from $1 < p < 2$ to $p = 2$ as show the following results:

Theorem 1.1. *Let M be a compact Riemannian manifold of dimension $n \geqslant 2$ and $\mathcal{M}$ the space of smooth Riemannian metrics on M. Assume $1 < p < \min\{2, \sqrt{n}\}$. Then, the map $g \in \mathcal{M} \mapsto B_0(p,g)$ is continuous in the C^0-topology, i.e. if the components of metric g_{ij}^α converges to g_{ij} in $C^0(M)$, then $B_0(p,g^\alpha) \to B_0(p,g)$ as $\alpha \to +\infty$.*

Theorem 1.2. *Let M be a compact Riemannian manifold of dimension n and $\mathcal{M}$ as in Theorem* 1.1. *Assume $p = 2$ and $n \geqslant 4$. If (g^α) is a sequence in $\mathcal{M}$ such that $g^\alpha \to g$ in $C^0(M)$ and $Scal_{g^\alpha} \to Scal_g$ pointwise in M, where $Scal_g$ denotes the scalar curvature of the metric g, then $B_0(2,g^\alpha) \to B_0(2,g)$ as $\alpha \to +\infty$. In particular, the map $g \in \mathcal{M} \mapsto B_0(2,g)$ is continuous in the C^2-topology. Moreover, the scalar curvature convergence or C^2-convergence assumption is necessary.*

The proof of Theorems 1.1 and 1.2 are made by contradiction. The proofs consist in finding estimates for a family of minimizers of geometry-dependent functionals around a concentration point. These ideas are inspired in the work of Djadli and Druet [4].

2. Proof of Theorems 1.1 and 1.2

We present a sketch of the proof of Theorem 1.2. Let (g_α) be a sequence of metrics on M such that g_α converges to a metric g in the C^0-topology and $Scal_{g^\alpha}$ converges to $Scal_g$ pointwise in M. Suppose, by contradiction, that there exists $\varepsilon_0 > 0$ such that $|B_0(2,g_\alpha) - B_0(2,g)| > \varepsilon_0$ for infinitely many α. Then, at least, one of the situations holds: $B_0(2,g) - B_0(2,g_\alpha) > \varepsilon_0$ or $B_0(2,g_\alpha) - B_0(2,g) > \varepsilon_0$ for infinitely many α. If the first situation holds, replacing $B_0(2,g_\alpha)$ by $B_0(2,g) - \varepsilon_0$ in the optimal inequality associated to the metric g_α and letting $\alpha \to +\infty$, we contradict the definition of $B_0(2,g)$.

Suppose then that the second situation holds, i.e. $B_0(2,g) + \varepsilon_0 < B_0(2,g_\alpha)$ for infinitely many α. For each α, consider the functional

$$J_\alpha(u) = \int_M |\nabla_{g_\alpha} u|^2\,\mathrm{d}v_{g_\alpha} + (B_0(2,g) + \varepsilon_0)K(n,2)^{-2}\int_M u^2\,\mathrm{d}v_{g_\alpha}$$

defined on $\Lambda_\alpha = \{u \in H_1^2(M)\colon \int_M |u|^{2^*}\,\mathrm{d}v_{g_\alpha} = 1\}$. From the definition of $B_0(2, g_\alpha)$, it follows directly that $\lambda_\alpha := \inf_{\Lambda_\alpha} J_\alpha(u) < K(n,2)^{-2}$. But this implies the existence of a non-negative minimizer $u_\alpha \in \Lambda_\alpha$ for λ_α. The Euler–Lagrange equation for u_α is then

$$-\Delta_{g_\alpha} u_\alpha + \big(B_0(2,g) + \varepsilon_0\big) K(n,2)^{-2} u_\alpha = \lambda_\alpha u_\alpha^{2^*-1}, \tag{E_α}$$

where $\Delta_{g_\alpha} = \mathrm{div}_{g_\alpha}(\nabla_{g_\alpha})$ is the Laplacian operator with respect to the metric g_α. By the standard elliptic theory, u_α belongs to $C^\infty(M)$ and $u_\alpha > 0$ on M. Our goal now is to study the sequence $(u_\alpha)_\alpha$ as $\alpha \to +\infty$. From the convergence $g_\alpha \to g$, it follows that $(u_\alpha)_\alpha$ is bounded in $H_1^2(M)$ with respect to the metric g. So, there exists $u \in H_1^2(M)$, $u \geqslant 0$, such that $u_\alpha \rightharpoonup u$ weakly in $H_1^2(M)$ and $\lambda_\alpha \to \lambda$ as $\alpha \to +\infty$, up to a subsequence. Moreover, by the Sobolev embedding compactness theorem, one easily finds

$$\int_M u_\alpha^q \,\mathrm{d}v_{g_\alpha} \to \int_M u^q \,\mathrm{d}v_g \tag{1}$$

for any $1 \leqslant q < 2^*$. So, letting $\alpha \to +\infty$ in Eq. (E_α), one concludes that u satisfies

$$\Delta_g u + \big(B_0(2,g) + \varepsilon_0\big) K(n,2)^{-2} u = \lambda u^{2^*-1}. \tag{E}$$

Assume that $u \not\equiv 0$. In this case, by ($J_{g,\mathrm{opt}}^2$) and (E), one has

$$\begin{aligned}\left(\int_M u^{2^*}\,\mathrm{d}v_g\right)^{2/2^*} &< K(n,2)^2 \int_M |\nabla_g u|^2\,\mathrm{d}v_g + \big(B_0(2,g)+\varepsilon_0\big)\int_M u^2\,\mathrm{d}v_g \\ &= K(n,2)^2 \lambda \int_M u^{2^*}\,\mathrm{d}v_g \leqslant \int_M u^{2^*}\,\mathrm{d}v_g,\end{aligned}$$

since $0 \leqslant \lambda \leqslant K(n,2)^{-2}$. This implies that $\int_M u^{2^*}\,\mathrm{d}v_g > 1$. But this inequality contradicts $\int_M u^{2^*}\,\mathrm{d}v_g \leqslant \liminf \int_M u_\alpha^{2^*}\,\mathrm{d}v_{g_\alpha} = 1$. We then assume that $u = 0$ on M and prove that this assumption leads to a contradiction. We assert, in this case, that $\lambda_\alpha \to K(n,2)^{-2}$ as $\alpha \to +\infty$. In fact, noting that $\int_M u_\alpha^{2^*}\,\mathrm{d}v_g \to 1$ since $u_\alpha \in \Lambda_\alpha$, and $\lim \int_M u_\alpha^2\,\mathrm{d}v_{g_\alpha} = 0$ by (1), letting $\alpha \to +\infty$ in the Sobolev inequality associated to the metric g, one finds $\liminf \int_M |\nabla_g u_\alpha|^2\,\mathrm{d}v_g \geqslant K(n,2)^{-2}$, so that $\liminf \int_M |\nabla_{g_\alpha} u_\alpha|^2\,\mathrm{d}v_{g_\alpha} \geqslant K(n,2)^{-2}$. Therefore, combining this last inequality with $\int_M |\nabla_{g_\alpha} u_\alpha|^2\,\mathrm{d}v_{g_\alpha} \leqslant \lambda_\alpha$, it follows directly that $\lambda = K(n,2)^{-2}$. Let $x_\alpha \in M$ be a maximum point of u_α, i.e $u_\alpha(x_\alpha) = \|u_\alpha\|_\infty$. Let $x_0 \in M$ be such that $x_\alpha \to x_0$, up to a subsequence.

We divide the proof into three stages. We next only mention each one of them.

First stage: For each $R > 0$, we have

$$\lim_{\alpha\to+\infty} \int_{B_{g_\alpha}(x_\alpha, R\mu_\alpha)} u_\alpha^{2^*}\,\mathrm{d}v_{g_\alpha} = 1 - \varepsilon_R \tag{2}$$

where $\mu_\alpha = \|u_\alpha\|_\infty^{-2^*/n}$ and $\varepsilon = \varepsilon(R) \to 0$ as $R \to +\infty$.

Second stage: There exist constants $c, \delta > 0$, independent of α, such that $d_{g_\alpha}(x, x_\alpha)^{n/2^*} u_\alpha(x) \leqslant c$ for all $x \in \overline{B}_{g_\alpha}(x_\alpha, \delta)$, where d_{g_α} stands for the distance with respect to the metric g_α.

Third stage: For any $\delta > 0$ small enough,

$$\lim_{\alpha\to+\infty} \frac{\int_{M\setminus B_{g_\alpha}(x_0,\delta)} u_\alpha^2\,\mathrm{d}v_{g_\alpha}}{\int_M u_\alpha^2\,\mathrm{d}v_{g_\alpha}} = 0. \tag{3}$$

The proof of the third stage relies on the first and second ones.

We now argue with the third stage in order to obtain a contradiction. Some possibly different positive constants independent of α will be denoted by c. Combining the local isoperimetric inequality of [7] and the co-area formula, as done recently in [3], for any $\varepsilon > 0$, we easily find $\delta_\varepsilon > 0$, independent of α, such that

$$\left(\int_M |u|^{2^*}\,\mathrm{d}v_{g_\alpha}\right)^{2/2^*} \leqslant K(n,2)^2 \int_M |\nabla_{g_\alpha} u|^2\,\mathrm{d}v_{g_\alpha} + B_\varepsilon(g_\alpha)\int_M u^2\,\mathrm{d}v_{g_\alpha} \tag{4}$$

for all $u \in C_0^\infty(B_{g_\alpha}(x_0,\delta_\varepsilon))$, where $B_\varepsilon(g_\alpha) = \frac{n-2}{4(n-1)} K(n,2)^2 (Scal_{g_\alpha}(x_0) + \varepsilon)$. Fix $0 < \varepsilon < \varepsilon_0$ and consider a smooth cutoff function η_α such that $0 \leqslant \eta_\alpha \leqslant 1$, $\eta_\alpha = 1$ in $B_{g_\alpha}(x_0, \delta_\varepsilon/4)$ and $\eta_\alpha = 0$ in $M \setminus B_{g_\alpha}(x_0, \delta_\varepsilon/2)$. Taking $u = \eta_\alpha u_\alpha$ in (4), using the identity

$$\int_M \left|\nabla_{g_\alpha}(\eta_\alpha u_\alpha)\right|^2 \mathrm{d}v_{g_\alpha} = -\int_M \eta_\alpha^2 u_\alpha \Delta_{g_\alpha} u_\alpha \, \mathrm{d}v_{g_\alpha} + \int_M |\nabla_{g_\alpha}\eta_\alpha|^2 u_\alpha^2 \, \mathrm{d}v_{g_\alpha},$$

Eq. (E_α) and the third stage, one arrives at

$$\left(\int_M |\eta_\alpha u_\alpha|^{2^*} \mathrm{d}v_{g_\alpha}\right)^{2/2^*} - \int_M \eta_\alpha^2 |u_\alpha|^{2^*} \mathrm{d}v_{g_\alpha} \leqslant -\big(B_0(g) + \varepsilon_0\big) \int_M \eta_\alpha^2 u_\alpha^2 \, \mathrm{d}v_{g_\alpha} + B_\varepsilon(g_\alpha) \int_M \eta_\alpha^2 u_\alpha^2 \, \mathrm{d}v_{g_\alpha} + c \int_M |\nabla_{g_\alpha}\eta_\alpha|^2 u_\alpha^2 \, \mathrm{d}v_{g_\alpha}.$$

By Hölder inequality,

$$\int_M \eta_\alpha^2 |u_\alpha|^{2^*} \mathrm{d}v_{g_\alpha} \leqslant \left(\int_M |\eta_\alpha u_\alpha|^{2^*} \mathrm{d}v_{g_\alpha}\right)^{2/2^*} \left(\int_M |u_\alpha|^{2^*} \mathrm{d}v_{g_\alpha}\right)^{(2^*-2)/2^*} \leqslant \left(\int_M |\eta_\alpha u_\alpha|^{2^*} \mathrm{d}v_{g_\alpha}\right)^{2/2^*},$$

so that

$$\big(B_0(g) - B_\varepsilon(g_\alpha) + \varepsilon_0\big) \int_M \eta^2 u_\alpha^2 \, \mathrm{d}v_{g_\alpha} \leqslant c \int_M |\nabla_{g_\alpha}\eta_\alpha|^2 u_\alpha^2 \, \mathrm{d}v_{g_\alpha}.$$

This inequality imply

$$\frac{n-2}{4(n-1)} K(n,2)^2 (Scal_g - Scal_{g_\alpha})(x_0) + \varepsilon_0 - \varepsilon \leqslant c \frac{\int_{M \setminus B_{g_\alpha}(x_0,\delta_\varepsilon/2)} u_\alpha^2 \, \mathrm{d}v_{g_\alpha}}{\int_M u_\alpha^2 \, \mathrm{d}v_{g_\alpha}}.$$

Letting $\alpha \to +\infty$ and applying again the third stage, we clearly find the desired contradiction. As claimed in the statement of Theorem 1.2, the scalar curvature convergence or the C^2-convergence assumption is necessary as shows the following example:

Let (M, g_0) be a smooth compact Riemannian manifold of dimension $n \geqslant 4$. Let $(f_\alpha)_\alpha \subset C^\infty(M)$ be a sequence of positive functions converging to the constant function $f_0 = 1$ in $L^p(M)$, $p > n$, such that $\max_M f_\alpha \to +\infty$. Let $u_\alpha \in C^\infty(M)$, $u_\alpha > 0$, be the solution of $-c(n)\Delta_{g_0} u + u = f_\alpha$, where $c(n) = \frac{4(n-1)}{n-2}$. By elliptic L^p-theory, it follows that $(u_\alpha)_\alpha$ is bounded in $H^{2,p}(M)$, so that u_α converges to u_0 in $C^{1,\beta}(M)$ for some $0 < \beta < 1$. Moreover, $u_0 = 1$ since f_α converges to 1 in $L^p(M)$ and the constant function 1 is the unique solution of the limit problem. In particular, $g_\alpha = u_\alpha^{2^*-2} g_0$ converges to g_0 only in the $C^{1,\beta}$-topology. In addition, it follows easily that $\max_M Scal_{g_\alpha} \to +\infty$, so that $B_0(2, g_\alpha) \to +\infty$. The proof of Theorem 1.1 follows closely the same ideas above and the final contradiction is obtained independent of any additional information on the scalar curvatures.

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