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C. R. Acad. Sci. Paris, Ser. I 346 (2008) 887–892

COMPTES RENDUS MATHEMATIQUE

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Probability Theory

Uniqueness of embedding of Gaussian probability measures into a continuous convolution semigroup on simply connected nilpotent Lie groups

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Received 17 November 2006; accepted after revision 22 October 2007

Available online 2 May 2008

Presented by Marc Yor

Abstract

Let $\{\mu_t^{(i)}\}_{t\geqslant 0}$ $(i=1,2)$ be continuous convolution semigroups on a simply connected nilpotent Lie group G. Suppose that $\mu_1^{(1)}=\mu_1^{(2)}$ and that $\{\mu_t^{(1)}\}_{t\geqslant 0}$ is a Gaussian semigroup (in the sense that its generating distribution just consists of a primitive distribution and a second order differential operator). Then $\mu_t^{(1)}=\mu_t^{(2)}$ for all $t\geqslant 0$. ***To cite this article: D. Neuenschwander, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Unicité du plongement de mesures de probabilité gaussiennes dans un semigroupe de convolution continu sur des groupes de Lie nilpotents et simplement connexes. Soient $\{\mu_t^{(i)}\}_{t\geqslant 0}$ $(i=1,2)$ des semigroupes de convolution continus sur un groupe de Lie G nilpotent et simplement connexe. Si $\mu_1^{(1)}=\mu_1^{(2)}$ et si $\{\mu_t^{(1)}\}_{t\geqslant 0}$ est un semigroupe gaussien (au sens que sa distribution génératrice ne consiste que d'une distribution primitive et d'un opérateur différentiel de second ordre), alors $\mu_t^{(1)}=\mu_t^{(2)}$ pour tout $t\geqslant 0$. ***Pour citer cet article : D. Neuenschwander, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Soit G un groupe de Lie nilpotent de degré r et simplement connexe. Il est bien connu (formule de Lévy–Hinčin) que la distribution génératrice $\mathcal{A}(f)=\frac{d}{dt}|_{t=0+}\int_G f(x)\mu_t(dx)$ d'un semigroupe de convolution continu de mesures de probabilité (s.c.c.) $\{\mu_t\}_{t\geqslant 0}$ sur G consiste d'une distribution primitive, d'un opérateur différentiel de second ordre et d'un générateur de Poisson généralisé. Le s.c.c. est appelé gaussien si le terme de Poisson généralisé disparait. Une

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2007.10.038

mesure de probabilité μ sur G est appelée gaussienne si elle est plongeable dans un s.c.c. gaussien $\{\mu_t\}_{t\geqslant 0}$ (c.-à-d. $\mu=\mu_1$). Pap [14] a démontré que si $\mathcal{S}^{(1)}=\{\mu_t^{(1)}\}_{t\geqslant 0}$ et $\mathcal{S}^{(2)}=\{\mu_t^{(2)}\}_{t\geqslant 0}$ sont des s.c.c. gaussiens et si $\mu_1^{(1)}=\mu_1^{(2)}$, alors $\mathcal{S}^{(1)}=\mathcal{S}^{(2)}$, mais il laissait ouvert la question si une mesure gaussienne sur G est peut-être aussi plongeable dans un s.c.c. non-gaussien sur G. Dans notre présente Note, nous prouvons que cela n'est vraiment pas possible et donc que chaque mesure gaussienne sur G n'est plongeable que dans un seul s.c.c. sur G.

La méthode de preuve est la suivante. Par le résultat de Pap [14] sus-mentionné, il suffit de prouver que $\mathcal{S}^{(2)}$ est aussi gaussien. Définissons les mesures non-négatives et bornées $\eta_t^{(i)}:=(1/t)\mu_t^{(i)}$ et posons $\kappa^{(i)}:=\int_G\|x\|^4\eta_t^{(i)}(dx)$. On peut démontrer que les queues de ces moments absolus d'ordre 4 des mesures $\eta_t^{(1)}$ sont uniformément intégrables pour $t\leqslant 1$. Par le « Théorème de l'approximation Poissonienne » sur G, l'identification de distributions génératrice sur G avec celles sur l'espace vectoriel sous-jacent et les conditions classique de convergence de lois infiniment divisibles sur un espace vectoriel de dimension finie, la gaussiennité de $\mathcal{S}^{(1)}$ implique que $\kappa_t^{(1)}\to 0$. La gaussiennité de $\mathcal{S}^{(1)}$ implique que $\kappa_t^{(1)}\to 0$ $(t\to 0)$. Done qu'aussi chaque mesure $\mu_{2^{-n}}^{(2)}$ possède tous les moments. Un calcul récursif (en utilisant la nilpotence) des moments implique que les mesures $\mu_{2^{-n}}^{(1)}$ et $\mu_{2^{-n}}^{(2)}$ ont (pour chaque n fixé) les mêmes moments, donc aussi $\kappa_{2^{-n}}^{(2)}\to 0$ $(n\to\infty)$.

1. Introduction

Let G be a locally compact group, e the neutral element, $G^*:=G\backslash\{e\}$. The structure $(M^1(G),*,\xrightarrow{w})$ is the topological semigroup of (regular) probability measures on G, equipped with the operation of convolution and the weak topology (cf. Heyer [9], Theorem 1.2.2). A continuous convolution semigroup $\{\mu_t\}_{t\geqslant 0}$ of probability measures on G (c.c.s. for short) is a continuous semigroup homomorphism

$$\big([0,\infty[,+\big)\ni t\mapsto \mu_t\in\big(M^1(G),*,\xrightarrow{w}\big),\qquad \mu_0=\varepsilon_e$$

(ε_x denoting the Dirac probability measure at $x\in G$). For simply connected nilpotent Lie groups the request $\mu_0=\varepsilon_e$ is no restriction, since in any case μ_0 has to be an idempotent element of $M^1(G)$ and is thus the Haar measure ω_K on some compact subgroup $K\subset G$ (cf. Heyer [9], 1.5.6); however, simply connected nilpotent Lie groups have no non-trivial compact subgroups (cf. Nobel [13], 2.2). Let G be a Lie group, $C_b^\infty(G)$ the space of bounded complex-valued C^∞-functions on G, and $\mathcal{D}(G)$ the subspace of complex-valued C^∞-functions with compact support.

The generating distribution $\mathcal{A}$ of a c.c.s. $\{\mu_t\}_{t\geqslant 0}$ is defined (for $f\in\mathcal{D}(G)$) as

$$\mathcal{A}(f):=\lim_{t\to 0+}\frac{1}{t}\int_G\big(f(x)-f(e)\big)\mu_t(dx)=\frac{d}{dt}\Big|_{t=0+}\int_G f(x)\mu_t(dx)\quad\big(f\in\mathcal{D}(G)\big).$$

It exists on the whole of $C_b^\infty(G)$ (cf. Siebert [18], p. 119). Now let G be a simply connected nilpotent Lie group. This means that G is a Lie group with Lie algebra $\mathcal{G}$ such that $\exp:\mathcal{G}\to G$ is a diffeomorphism and that the descending central series is finite, i.e. there is some $r\in\mathbb{N}_0$ such that $\mathcal{G}_0\supsetneq\mathcal{G}_1\supsetneq\cdots\supsetneq\mathcal{G}_r=\{0\}$, where $\mathcal{G}_0:=\mathcal{G}$, $\mathcal{G}_{k+1}:=[\mathcal{G},\mathcal{G}_k]$ $(0\leqslant k\leqslant r-1)$. G is then called step r-nilpotent. We may identify G with $\mathcal{G}=\mathbb{R}^d$ via log (the inverse map of exp). So G may be interpreted as $\mathbb{R}^d$ equipped with a Lie bracket $[\cdot,\cdot]:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ which is bilinear, skew-symmetric, and satisfies the Jacobi identity

$$\big[[x,y],z\big]+\big[[y,z],x\big]+\big[[z,x],y\big]=0.$$

The group product is then given by the Campbell–Hausdorff formula (cf. Serre [16], Neuenschwander [10], p. 9), where due to the nilpotency only the terms up to order r arise.

The first few terms are

$$x\cdot y=x+y+\frac{1}{2}[x,y]+\frac{1}{12}\big(\big[[x,y],y\big]+\big[[y,x],x\big]\big)+\cdots.$$

It is clear that the neutral element e is 0 and that the inverse element of x is given by $-x$. The generating distribution of a c.c.s. on G assumes a very explicit form: The functional $\mathcal{A}$ on $C_b^\infty(G)$ is a generating distribution of a c.c.s. $\{\mu_t\}_{t\geqslant 0}$ iff it has the form (Lévy–Hinčin formula)

$$\mathcal{A}(f)=\langle\xi,\nabla\rangle f(0)+\frac{1}{2}\langle\nabla,M\cdot\nabla\rangle f(0)+\int_{G^*}\big[f(x)-f(0)-\Psi(f,x)\big]\eta(dx),$$

where

$$\Psi(f,x) := \begin{cases} \langle x, \nabla\rangle f(0)\colon\ \|x\| \leqslant 1, \\ \langle \frac{x}{\|x\|}, \nabla\rangle f(0)\colon\ \|x\| > 1 \end{cases}$$

($f \in C_b^\infty(G)$), $\xi \in G \cong \mathcal{G} \cong \mathbb{R}^d$, M is a positive semidefinite $d \times d$-matrix, and η is a Lévy measure on G^*, i.e. a non-negative measure on G^* satisfying

$$\int\limits_{0<\|x\|\leqslant 1} \|x\|^2 \eta(dx) + \eta\big(\{x \in G\colon\ \|x\| > 1\}\big) < \infty.$$

ξ, M, η are uniquely determined by $\{\mu_t\}_{t\geqslant 0}$. (Cf. Siebert [17], Satz 1.) As a shorthand we will write $\mathcal{A} = [\xi, M, \eta]$. In particular, we have that $(1/t)\mu_t|_B \overset{w}{\to} \eta|_B$ ($t \to 0$) for every open subset B of G bounded away from 0 (this follows immediately from the definition of the generating distribution). The generating distribution $\mathcal{A}$ on $C_b^\infty(G)$ determines uniquely the c.c.s. $\{\mu_t\}_{t\geqslant 0}$ (cf. Siebert [17], Satz 1), for this reason we may write $\mu_t =: \operatorname{Exp} t\mathcal{A}$ ($t \geqslant 0$). The generating distribution $\mathcal{A}$ is called primitive if it is of the form $[\xi, 0, 0]$. It is called Gaussian if $\eta = 0$. The c.c.s. $\{\operatorname{Exp} t\mathcal{A}\}_{t\geqslant 0}$ is called Gaussian if $\mathcal{A}$ is Gaussian. A measure $\mu \in M^1(G)$ is called Gaussian if $\mu = \mu_1$ for some Gaussian c.c.s. $\{\mu_t\}_{t\geqslant 0}$. A G-valued random variable X is called Gaussian if its law $\mathcal{L}(X) \in M^1(G)$ is Gaussian.

It has been shown by Burrell, McCrudden [3] that every infinitely divisible probability measure on a simply connected nilpotent Lie group G is embeddable into a c.c.s. on G (as it is well known for finite-dimensional vector spaces). But now also the question of uniqueness of the embedding c.c.s. is of great importance. If an infinitely divisible law on G has exactly one embedding c.c.s., then convergence of c.c.s. at time $t = 1$ to this measure implies automatically convergence of the corresponding c.c.s. (i.e. for all time points $t \geqslant 0$) to the limit c.c.s. By the 'Accompanying Poisson Laws Theorem' for triangular arrays of rowwise identical probability measures on the simply connected nilpotent Lie group G it follows furthermore that if μ_1 is embeddable into a unique c.c.s. $\{\mu_t\}_{t\geqslant 0}$, then, for a strictly increasing sequence $\{k(n)\}_{n\geqslant 1}$ of natural numbers and for a sequence $\{\nu_n\}_{n\geqslant 1} \subset M^1(G)$, the relation $\nu_n^{*k(n)} \overset{w}{\to} \mu_1$ ($n \to \infty$) implies $\nu_n^{*\lfloor k(n)t\rfloor} \overset{w}{\to} \mu_t$ ($n \to \infty$) ($t \geqslant 0$) (cf. Nobel [13], Remark 2(a)). It is well known that this uniqueness property is true for $(\mathbb{R}^d, +)$. Finite groups satisfy the uniqueness property for their c.c.s. iff every non-neutral element has order 2 (then the group is of course Abelian) (cf. Böge [2]). For locally compact Abelian groups, a sufficient condition for the uniqueness property is the request that the group has no non-trivial compact subgroup (cf. Heyer [9], Theorem 3.5.15). For irreducible symmetric spaces G/K of non-compact type (i.e. G a semisimple non-compact Lie group with finite center and K a maximal compact subgroup) and K-biinvariant probability measures μ on G Graczyk [5] used a method to associate to μ a bounded non-negative measure $\check{\mu}$ on a Cartan subalgebra $(\mathbf{a}, +)$ such that $\mu_1 * \mu_2 = \mu_3$ iff $\check{\mu}_1 * \check{\mu}_2 = \check{\mu}_3$ and such that $\check{\mu}$ determines μ uniquely. This readily yields the uniqueness property for all c.c.s. of K-biinvariant probability measures on G by the uniqueness property on $(\mathbf{a}, +)$. In more general framework, some partial results have been obtained by Hazod [6]. For stable and semistable semigroups on simply connected nilpotent Lie groups see Nobel [13], Hazod, Scheffler [7], and Hazod, Siebert [8]. A partial result for Poisson semigroups on simply connected nilpotent Lie groups has been obtained in Neuenschwander [11]. Pap [14] proved the uniqueness property for the Gaussian semigroups among all Gaussian semigroups on simply connected nilpotent Lie groups, generalizing the corresponding result for simply connected step 2-nilpotent Lie groups by Baldi [1], but he left open the question if Gaussian measures can also be embedded into non-Gaussian c.c.s. In Neuenschwander [10] it was shown that on the three-dimensional Heisenberg group, this is indeed not the case, i.e. every Gaussian probability measure on the three-dimensional Heisenberg group is embeddable into a unique c.c.s. The generalization of this result to all simply connected step 2-nilpotent Lie groups can be found in Neuenschwander [12]. For the more general framework of nilpotent quantum groups and braided groups see Franz, Neuenschwander, Schott [4].

In the present Note, we will show that this uniqueness property of the embedding c.c.s. of a Gaussian measure among all c.c.s. is indeed true for simply connected Lie groups G which are nilpotent of any step r. The general idea of proof will be a recursive calculation of moments in order to show that all embedding c.c.s. of a Gaussian probability measure must be Gaussian. Then the assertion follows from Pap's afore-mentioned uniqueness result [14] among all Gaussian c.c.s.

2. The result and its proof

Theorem 1. *Let $\{\mu_t^{(i)}\}_{t\geqslant 0}$ $(i=1,2)$ be c.c.s. on the simply connected nilpotent Lie group G. Suppose that $\mu_1^{(1)}=\mu_1^{(2)}$ and that $\{\mu_t^{(1)}\}_{t\geqslant 0}$ is Gaussian. Then $\mu_t^{(1)}=\mu_t^{(2)}$ for all $t\geqslant 0$.*

One of the main ingredients of the proof will be the following property of recursive calculability of moments of convolution roots (cf. Neuenschwander [11]):

Assume G is simply connected and step r-nilpotent. Consider an adapted vector space decomposition of $G=\mathcal{G}$, i.e.

$$G=\mathcal{G}=\mathbb{R}^d=\bigoplus_{i=1}^{r} V_i$$

such that $\bigoplus_{i=k}^{r} V_i=\mathcal{G}_{k-1}$, where $\{\mathcal{G}_k\}_{0\leqslant k\leqslant r}$ is the descending central series:

$$\mathcal{G}_0:=\mathcal{G},\qquad \mathcal{G}_{k+1}:=[\mathcal{G},\mathcal{G}_k]$$

(and thus $\mathcal{G}_r=\{0\}$). In this case, one can take a Jordan–Hölder basis for $\mathcal{G}=\mathbb{R}^d$, i.e. a basis $E=\{e_1,e_2,\ldots,e_d\}=\bigcup_{i=1}^{r} E_i$ where $E_i=\{e_{i,1},e_{i,2},\ldots,e_{i,d(i)}\}$ is a basis of V_i ($d(i)$ thus being the dimension of V_i).

Take, on $\mathbb{N}_0^d$, the lexicographic ordering from behind defined as follows: Put $(a_1,a_2,\ldots,a_d)<(b_1,b_2,\ldots,b_d)$ if $(a_d,a_{d-1},\ldots,a_{d-j+1})=(b_d,b_{d-1},\ldots,b_{d-j+1})$ and if $a_{d-j}<b_{d-j}$ for some $j\in\{0,1,\ldots,d-1\}$.

Consider the component representation $G\ni x=:\sum_{j=1}^{d} x_j e_j$. For $\mu\in M^1(G)$, $\ell=(\ell_1,\ell_2,\ldots,\ell_d)\in\mathbb{N}_0^d$, define the mixed moments

$$M_\ell(\mu):=\int_G \prod_{j=1}^{d} x_j^{\ell_j}\,\mu(dx)$$

(if they exist).

Lemma 1. *Assume μ, ν are probability measures on the simply connected nilpotent Lie group G satisfying $\mu=\nu*\nu$ and such that all moments $M_\ell(\mu)$ $(\ell\in\mathbb{N}_0^d)$ exist absolutely. Then all $M_\ell(\nu)$ exist absolutely and the $M_\ell(\nu)$ $(\ell\in\mathbb{N}_0^d)$ may be calculated out of the $M_\ell(\mu)$ recursively with respect to ℓ.*

Proof. Since μ possesses all absolute moments, it is clear (by the group property of G) that ν cannot have a qualitatively heavier tail behavior (in the sense of absolute integrability of power functions) than μ itself, thus must possess all absolute moments, too. Assume X,Y are i.i.d. G-valued random variables with law $\mathcal{L}(X)=\nu$. Write

$$M_\ell(\mu)=E\left(\prod_{j=1}^{d}(X\cdot Y)_j^{\ell_j}\right)=E\left(\prod_{j=1}^{d}\left(X+Y+\frac{1}{2}[X,Y]+\cdots\right)_j^{\ell_j}\right).$$

By the adaptedness, we get, by multiplying out and considering the components

$$\left(X+Y+\frac{1}{2}[X,Y]+\cdots\right)_j^{\ell_j}=X_j^{\ell_j}+Y_j^{\ell_j}+P_j,$$

P_j being a polynomial in $X_1,Y_1,X_2,Y_2,\ldots,X_j,Y_j$, where in every monomial the exponents of X_j and Y_j are strictly smaller than ℓ_j. Now, by multiplying out the product $\prod_{j=1}^{d}(\ldots)_j^{\ell_j}$, we get

$$\prod_{j=1}^{d}(\ldots)_j^{\ell_j}=\prod_{j=1}^{d}X_j^{\ell_j}+\prod_{j=1}^{d}Y_j^{\ell_j}+P,$$

where P is a polynomial in $X_1,Y_1,X_2,Y_2,\ldots,X_d,Y_d$ such that for every monomial $\prod_{j=1}^{d}(X_j^{r_j}Y_j^{s_j})$ we have $(r_1,r_2,\ldots,r_d),(s_1,s_2,\ldots,s_d)<\ell$. Now the assertion follows from the independence of X and Y and the fact that $E(\prod_{j=1}^{d}X_j^{\ell_j})=E(\prod_{j=1}^{d}Y_j^{\ell_j})$. □

We will use the 'Poisson Approximation Theorem' (or 'Accompanying Laws Theorem') in the following form (cf. e.g. Neuenschwander [10], Chapter 1, Proposition 1.3):

Lemma 2. *Let $\{\mu_t\}_{t\geqslant 0}$ be a c.c.s. on the simply connected nilpotent Lie group G. Then we have*

$$\exp\big((s/t)(\mu_t-\varepsilon_0)\big)\xrightarrow{w}\mu_s\quad (t\to 0)$$

for all $s\geqslant 0$.

(*Here* exp *denotes the exponential power series in the Banach algebra of bounded signed measures on G.*)

Together with Neuenschwander [10], Chapter 1, Proposition 1.2, the identification of generating distributions on simply connected nilpotent Lie groups with those on the underlying Lie algebra, and the well known convergence conditions for infinitely divisible laws on finite-dimensional vector spaces implies Lemma 3:

Lemma 3. *Assume $\{\mu_t\}_{t\geqslant 0}$ is a c.c.s. on the simply connected nilpotent Lie group G with generating distribution $[\xi, M, \eta]$. Let $m_{i,i}$ be the diagonal elements of M. Then*

$$2^n\mu_{2^{-n}}(B)\to\eta(B)\quad (n\to\infty)$$

(*for every Borel subset B of G bounded away from* 0 *and carrying η-measure zero on its boundary*) *and*

$$\limsup_{n\to\infty} 2^n\int\limits_{\{x\in G:\ \|x\|\leqslant\varepsilon\}} x_i^2\mu_{2^{-n}}(dx)\to m_{i,i}\quad (\varepsilon\to 0).$$

The symbol C will be used for a generic positive finite constant (depending only on the fixed step r of nilpotency) of possibly changing value.

Proof of Theorem 1. Let $\mathcal{S}^{(i)}=\{\mu_t^{(i)}\}_{t\geqslant 0}$ ($i=1,2$) be c.c.s. on the simply connected step r-nilpotent Lie group G. Assume that $\mathcal{S}^{(1)}$ is Gaussian and that $\mu_1^{(1)}=\mu_1^{(2)}$. By the above-mentioned uniqueness result for the embedding Gaussian c.c.s. of a Gaussian measure due to Pap [14] it suffices to show that also $\mathcal{S}^{(2)}$ is Gaussian. Define the bounded non-negative measures $\eta_t^{(i)}:=(1/t)\mu_t^{(i)}$ and put

$$\kappa_t^{(i)}:=\int\limits_G\|x\|^4\eta_t^{(i)}(dx)\in[0,\infty].$$

We first want to show that the fourth absolute moments of $\eta_t^{(1)}$ are uniformly integrable for $0\leqslant t\leqslant 1$. Let Z be an iterated stochastic integral of m ($m\leqslant r$) one-dimensional standard Brownian motions:

$$Z=\int\limits_0^1\int\limits_0^{s_m}\int\limits_0^{s_{m-1}}\cdots\int\limits_0^{s_2} dB_1(s_1)\,dB_2(s_2)\cdots dB_m(s_m).$$

Suppose that for every pair of coordinates the Brownian motions are either independent or identical. If we assume that all occurring one-dimensional Brownian motions are independent, we can directly invoke a result in Schott [15] giving the asymptotic behavior of the density of Z in this case and which implies the tail estimate

$$P\big(|Z|\geqslant x\big)=\mathrm{O}\big(\exp(-Cx^{2/m})\big)\quad (x\to\infty).$$

In case that identical Brownian motions occur, by using the substitution $dB_i\,dB_i=dt$, one sees easily that the additional terms (in contrast to the situation with independent Brownian motions) in the evaluation of the stochastic integrals do not disturb the qualitative tail behavior.

Since multi-dimensional centered Brownian motions on the vector space $(\mathbb{R}^d,+)$ can always be written as a linear transformation of a multi-dimensional standard Brownian motion, it follows that every component of a Gaussian random variable on G has the form of a linear combination of iterated stochastic integrals of the just-mentioned type (with $m\leqslant r$ independent or identical one-dimensional standard Brownian motions), thus has the tail-behavior

$O(\exp(-C|x|^{2/r}))$ as $|x| \to \infty$. By the scaling property (self-similarity) of the one-dimensional standard Brownian motion, one sees easily that this "exponential" tail decrease $O(\exp(-C|x|^{2/r}))$ holds uniformly in time $t \in [0,1]$. Thus the Gaussianity of $\mathcal{S}^{(1)}$ and Lemma 3 imply that $\kappa^{(1)}_{2^{-n}} \to 0$ ($n \to \infty$). By Lemma 1 also the measures $\mu^{(2)}_{2^{-n}}$ possess all absolute moments and moreover, for every fixed n and ℓ, we have that $M_\ell(\mu^{(1)}_{2^{-n}}) = M_\ell(\mu^{(2)}_{2^{-n}})$. So it follows that also $\kappa^{(2)}_{2^{-n}} \to 0$ ($n \to \infty$), which implies (by Lemma 3) that indeed also $\mathcal{S}^{(2)}$ must be Gaussian. $\square$

Acknowledgements

The author wishes to thank Sara Fischer, Samuel Gutmann, and Dieter Ryter for their help in preparing the manuscript.

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