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Harmonic Analysis/Mathematical Analysis

Fourier restriction, polynomial curves and a geometric inequality

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Abstract

We announce a Fourier restriction result for general polynomial curves in $\mathbb{R}^d$. Measuring the Fourier restriction with respect to the affine arclength measure of the curve, we obtain a universal bound for the class of all polynomial curves of bounded degree. Our method relies on establishing a geometric inequality for general polynomial curves which is of interest in its own right. There are applications of this geometric inequality to other problems in euclidean harmonic analysis. ***To cite this article: S. Dendrinos, J. Wright, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Restrictions de Fourier et courbes polynomiales ; une inégalité géométrique. Le résultat que nous annonçons sur les restrictions de Fourier vaut pour des courbes polynomiales générales dans $\mathbb{R}^d$. Il permet de contrôler la norme L^q de la transformée de Fourier relativement à la mesure d'arc affine (dont nous rappelons la définition) à la norme L^p de la fonction, pour des p et q convenables. La borne est universelle pour toutes les courbes polynomiales de degré donné. Notre méthode repose sur une inégalité géométrique concernant les courbes polynomiales qui est intéressante en elle même, et s'applique à d'autres problèmes d'analyse harmonique euclidienne. ***Pour citer cet article : S. Dendrinos, J. Wright, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

Recently there has been considerable attention given to certain euclidean harmonic analysis problems associated to a surface or curve (for example, the problems of Fourier restriction and the smoothing effects of generalised Radon transforms) where the underlying surface measure is replaced by the so-called affine arclength measure. See [1,3,4,6, 8,10–15] and [16]. This has the effect of making the problem affine invariant as well as invariant under reparametrisations of the underlying surface. For this reason there have been many attempts to obtain universal results, establishing uniform bounds over a large class of surfaces or curves. The affine arclength measure also has the mitigating effect of dampening any curvature degeneracies of the variety and therefore one expects that the universal bounds one seeks will be the same as those arising from the most non-degenerate situation.

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In this article we announce such a result for the problem of Fourier restriction to a general polynomial curve in $\mathbb{R}^d$. More specifically, if $\Gamma : I \to \mathbb{R}^d$ parametrises a smooth curve in $\mathbb{R}^d$ on an interval I, set

$$L_\Gamma(t) = \det\big(\Gamma'(t) \cdots \Gamma^{(d)}(t)\big);$$

this is the determinant of a $d \times d$ matrix whose jth column is given by the jth derivative of Γ, $\Gamma^{(j)}(t)$. The affine arclength measure $\nu = \nu_\Gamma$ on Γ is defined on a test function ϕ by

$$\nu(\phi) = \int\limits_I \phi\big(\Gamma(t)\big)\big|L_\Gamma(t)\big|^{\frac{2}{d(d+1)}}\,\mathrm{d}t;$$

one easily checks that this measure is invariant under reparametrisations of Γ.

A basic problem in the theory of Fourier restriction is to determine the exponents p and q so that the a priori estimate

$$\|\hat{f}|_\Gamma\|_{L^q(\Gamma,d\nu)} \leqslant C\|f\|_{L^p(\mathbb{R}^d)} \tag{1}$$

holds uniformly for a large class of curves Γ. This problem was first considered by Sjölin in [17] where he showed that (1) holds uniformly over all smooth convex curves in the place if and only if $p' = 3q$ (here p' denotes the conjugate exponent to p; $p' = p/(p-1)$ and $1 \leqslant p < 4/3$. See also [14]. The convexity assumption implies that $L_\Gamma(t)$ remains single-signed and Sjölin produced a plane curve Γ where L_Γ rapidly changes sign and (1) fails for any $p' = 3q$ and $1 < p < 4/3$ (Sjölin's argument establishing (1) for convex curves works for any smooth plane curve as long as the number of sign changes of L_Γ remains bounded).

By considering the non-degenerate example $\Gamma(t) = (t, t^2, \ldots, t^d)$ where $L_\Gamma \equiv$ constant, one sees that in order for (1) to hold with a uniform constant C independent of the interval I where $\Gamma : I \to \mathbb{R}^d$, we must have $p' = \frac{d(d+1)}{2}q$ and $1 \leqslant p < \frac{d^2+d+2}{d^2+d}$. The former restriction follows from a simple scaling argument whereas the latter restriction follows from work of Arkipov, Kuratsuba and Chubarikov, [2]. Furthermore, Drury [9] showed that these restrictions on p and q are sufficient for (1) to hold for this non-degenerate example (see also the recent work of Bak, Oberlin and Seeger [3]). We note here that on the critical line $p' = \frac{d(d+1)}{2}q$, (1) becomes affine invariant; that is, (1) remains unchanged if we consider any affine transformation of Γ.

In higher dimensions the problem of understanding when (1) holds was first considered by Drury and Marshall [11] (see also [10] and [12]). Recently Bak, Oberlin and Seeger [3] have shown that if $p' = \frac{d(d+1)}{2}$ and $1 \leqslant p < \frac{d^2+d+2}{d^2+d}$, then (1) holds for $\Gamma(t) = (t^{a_1}, \ldots, t^{a_d})$ where $a_1, \ldots, a_d$ are any real numbers and the constant C may be taken to depend only on p and d; in particular, it may be taken to be independent of the powers $(a_1, \ldots, a_d)$. Our purpose here is to initiate an extension of the theory to polynomial curves $\Gamma(t) = (P_1(t), \ldots, P_d(t))$ where each component P_j is a general real polynomial. We consider the curve as parametrised over the entire real line.

Theorem 1.1. *The inequality (1) holds for all polynomial curves of bounded degree if*

$$p' = \frac{d(d+1)}{2}q \quad \textit{and} \quad 1 \leqslant p < \frac{d^2+2d}{d^2+2d-d}.$$

The constant C may be taken to depend only on p, d and the degrees of the polynomials defining Γ.

Remarks 1.2.

- We expect Theorem 1.1 to remain true in the larger range $1 \leqslant p < \frac{d^2+d+2}{d^2+d}$.
- By considering the class of polynomial curves with bounded degree, we control the number of sign changes of L_Γ which seems natural in light of Sjölin's counterexample.

2. Outline of proof

By following M. Christ's argument in [7] which establishes (1) for the non-degenerate case $\Gamma(t) = (t, t^2, \ldots, t^d)$ in the range $1 \leqslant p < \frac{d^2+2d}{d^2+2d-2}$, matters are reduced to establishing two properties about

$$\Phi_\Gamma(t_1, \ldots, t_d) = \Gamma(t_1) + \cdots + \Gamma(t_d):$$

(a) Φ_Γ is 1-1;
(b) $|J_{\Phi_\Gamma}(t_1,\dots,t_d)| \geqslant C\prod_{j=1}^d |L_\Gamma(t_j)|^{\frac{1}{d}}\prod_{j<k}|t_j - t_k|$

where $J_{\Phi_\Gamma}(t_1,\dots,t_d) = \det(\Gamma'(t_1)\cdots\Gamma'(t_d))$ is the determinant of the Jacobian matrix for the mapping Φ_Γ. Even in the non-degenerate case $\Gamma(t) = (t, t^2, \dots, t^d)$, Φ_Γ is not quite 1-1 but it is $d!$ to 1 off a set of measure zero. Furthermore, in this case, the geometric inequality (b) is an equality.

For polynomial curves both (a) and (b) are false in general. However we will find a decomposition of $\mathbb{R} = \bigcup I$ into a bounded number (depending only on d and the degrees of the polynomials defining Γ) of disjoint intervals so that on the interior of each I^d, Φ_Γ is $d!$ to 1 off a set of measure zero and the geometric inequality (b) holds. Therefore by restricting the original operator to each I and applying Christ's argument, we obtain a proof of the theorem.

The decomposition is produced in two stages. The first stage produces an elementary decomposition of $\mathbb{R} = \bigcup J$ so that on each interval J, various polynomial quantities (more precisely, certain determinants of minors of the $d\times d$ matrix $(\Gamma'(t)\cdots\Gamma^{(d)})$, including L_Γ) are single-signed. This allows us to write down a formula relating J_{Φ_Γ} and L_Γ. When $d = 2$ this formula is particularly simple; namely,

$$J_{\Phi_\Gamma}(s,t) = P_1'(s)P_1'(t)\int_s^t \frac{L_\Gamma(w)}{P_1'(w)^2}\,\mathrm{d}w$$

for any $s, t \in J$ (here $\Gamma = (P_1, P_2)$). From this, using an argument of J. Steinig [18], one can establish the injectivity of Φ_Γ on $\{(t_1,\dots,t_d) \in J^d\colon\ t_1 < \cdots < t_d\}$; however simple examples show that (b) can fail on some J and therefore we need to decompose each $J = \bigcup I$ further so that on each I, (b) holds. More precisely, we have

$$\bigl|J_{\Phi_\Gamma}(t_1,\dots,t_d)\bigr| \geqslant C\prod_{j=1}^d \bigl|L_\Gamma(t_j)\bigr|^{\frac{1}{d}}\prod_{j<k}|t_j - t_k|$$

for all $(t_1,\dots,t_d) \in I^d$ where C depends only on d and the degrees of the polynomials defining Γ. This is the geometric inequality referred to above which has already found applications in the theory of generalised Radon transforms.

This second stage decomposition $J = \bigcup I$ is much more technical and derived from a certain algorithm which uses two further decomposition procedures generated by individual polynomials; one of these decomposition procedures has been used in other problems and first appeared in [5]. The algorithm exploits in a crucial way the affine invariance of the inequality (b); that is, the inequality is left invariant when Γ is replaced by $A\Gamma$ for any invertible $d\times d$ matrix A.

Details of the proof will appear elsewhere.

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