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Calculus of Variations

Asymptotic analysis of periodically-perforated nonlinear media at and close to the critical exponent

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Abstract

We give a general Γ-convergence result for vector-valued nonlinear energies defined on perforated domains for integrands with p-growth in the critical case $p = n$. We characterize the limit extra term by a formula of homogenization type. We also prove that for p close to n there are three regimes, two with a nontrivial size of the perforation (exponential and mixed polynomial-exponential), and one where the Γ-limit is always trivial. ***To cite this article: A. Braides, L. Sigalotti, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Analyse asymptotique dans l'étude de milieux perforés au voisinage d'un exposant critique. On établit un résultat général de Γ-convergence d'énergies vectorielles nonlinéares définies sur des domaines perforés, dans le cas où l'intégrande est de croissance p, dans le cas critique $p = n$; la limite est caractérisée par une formule de type homogénéisation. On démontre également que pour p voisin de n trois régimes sont possibles, deux avec une taille du perforation non triviale (exponentielle et polynomiale-exponentielle), et une taille pour laquelle la Γ-limite est toujours triviale. ***Pour citer cet article : A. Braides, L. Sigalotti, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

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On analyse le comportement asymptotique de fonctionnelles non linéaires F_δ avec intégrandes du croissance $p > 1$, définies sur des fonctions vectorielles satisfaisant des conditions aux limites de type Dirichlet dans des milieux périodiquement perforés (voir (4)). Ces milieux sont caractérisés par leur périodicité δ et par la taille des perforations $\varepsilon = \varepsilon(\delta)$ (voir (3)). Dans la terminologie de [5] notre résultat peut être traduit par l'équivalence, dans le cas $p = n$, des fonctionnelles F_δ et à des fonctionnelles :

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$$G_\delta(u)=\int_\Omega f(Du)\,\mathrm{d}x+\frac{|\log\varepsilon|^{n-1}}{\delta^n}\int_\Omega \varphi(u)\,\mathrm{d}x,\quad u\in W_0^{1,n}(\Omega;\mathbb{R}^m). \tag{1}$$

Notre méthode démontre que le régime exponentiel des perforations dérive de l'invariance par changement d'échelles de problèmes que caractérise la fonction φ (voir (9)) et de comportement logarithmique de leurs minimiseurs. La formule « capacitaire » pour φ, dans le cas $p<n$, doit être remplacée par une formule intéressante de type « homogénéisation », qui montre que l'énergie des perforations ne se concentre pas à la même échelle que celle des perforations elles-mêmes, de manière analogue aux fonctionnelles du type Ginzburg–Landau. Nous avons étendu notre analyse aux exposants p variables, et on a montre que le régime exponentiel s'étend jusqu'à $p-n=\mathrm{O}(\delta^{n/n-1})$, avec deux autres régimes possibles. Dans le cas $f(\xi)=|\xi|^p$ ce résultat peut être traduit par l'équivalence des fonctionnelles F_δ^p (voir (11)) et

$$G_\delta^p(u)=\int_\Omega |Du|^p\,\mathrm{d}x+C_p\frac{\varepsilon^{n-p}}{\delta^n}\left(\frac{1-\varepsilon^{n-p/p-1}}{n-p}\right)^{1-p}\int_\Omega |u|^p\,\mathrm{d}x,\quad u\in W_0^{1,p}(\Omega;\mathbb{R}^m) \tag{2}$$

(C_p calculable explicitement), valable pour tout p, et donnant des perforations polynomials pour $p<n$.

1. Introduction

An interesting and much studied class of problems are variational problems defined on varying domains. The prototype of these domains are *perforated domains*; i.e., those obtained from a fixed Ω by removing some periodic set, the simplest of which is a periodic array of closed sets:

$$\Omega_\delta=\Omega\setminus\bigcup_{i\in\mathbb{Z}^n}(\delta i+\varepsilon K), \tag{3}$$

with $\varepsilon=\varepsilon(\delta)$. On the set K, we suppose that it is a bounded closed set with nonempty interior. When we consider Dirichlet boundary conditions on the boundary of Ω_δ (or on the boundary of Ω_δ interior to Ω) the asymptotic behaviour of such problems is obtained by studying the Γ-convergence (see [4]) of the functionals:

$$F_\delta(u)=\begin{cases}\displaystyle\int_\Omega f(Du)\,\mathrm{d}x & \text{if } u\in W_0^{1,p}(\Omega;\mathbb{R}^m) \text{ and } u=0 \text{ on } \Omega\setminus\Omega_\delta,\\ +\infty & \text{otherwise,}\end{cases} \tag{4}$$

where f is an energy density growing as $|Du|^p$. Taking $f(Du)=|Du|^p$ above, we encounter some by-now 'classical' results, first observed by Marchenko and Khruslov [7], and subsequently recast in a variational setting by Cioranescu and Murat [6], in which case for a *nontrivial scaling* of the perforation the Γ-limit contains an additional 'strange term' in place of the internal boundary conditions. To obtain this form of the Γ-limit different choices of ε must be made according to the space dimension n, that in this case are:

$$\text{(polynomial scaling)}\quad \varepsilon=R\delta^{n/n-p}\quad \text{if } p<n \text{ (with } R>0), \tag{5}$$

$$\text{(exponential scaling)}\quad \varepsilon=\exp(-a/\delta^{n/n-1})\quad \text{if } p=n \text{ (with } a>0). \tag{6}$$

A complete analysis by Γ-convergence for energies with a general (quasiconvex) integrand f with p-growth, and depending on vector-valued functions has been performed by Ansini and Braides in the case leading to the polynomial scaling ($p<n$) [2]. In this paper we treat the case $p=n$, which is the one leading to the exponential scaling, by first giving general convergence result for this critical case, and then exploring the case when p is varying and close to n.

2. Asymptotic behaviour at the critical scaling

In the case $p=n$ we have the following general convergence result:

Theorem 2.1 *(Asymptotic behaviour at the critical exponent). Let $f:\mathbb{M}^{m\times n}\to[0,\infty)$ be a quasiconvex function with $f(0)=0$; we suppose that there exist $c_1,c_2,k>0$ such that*

$$c_1|A|^n\leqslant f(A)\leqslant c_2|A|^n,\qquad \bigl|f(A)-f(B)\bigr|\leqslant k|A-B|\,|A|^{n-1}+|B|^{n-1}|$$

for all $A, B \in \mathbb{M}^{m\times n}$. Let δ_j be a positive infinitesimal sequence and let $a > 0$. Then, upon passing to a subsequence of (δ_j) (not relabelled) and having set $T_j = \exp(a\delta_j^{-n/n-1})$, the limit

$$\varphi(z) = \sup_{s>0} \lim_{j\to\infty} \frac{(\log T_j)^{n-1}}{a^{n-1}} \min\left\{ \int_{B_{s\delta_j T_j}} \frac{f(T_j Du)}{T_j^n}\,\mathrm{d}x\colon\ u \in z + W_0^{1,n}(B_{s\delta_j T_j};\mathbb{R}^m),\ u = 0 \text{ on } K \right\} \tag{7}$$

exists for all z, and the functionals F_{δ_j} defined in (4) Γ-converge (with respect to the strong convergence of $L^n(\Omega;\mathbb{R}^m)$) to the functional $F_0 : L^n(\Omega;\mathbb{R}^m) \to [0+\infty)$ defined by:

$$F_0(u) = \begin{cases} \int_\Omega f(Du)\,\mathrm{d}x + \int_\Omega \varphi(u)\,\mathrm{d}x & \text{if } u \in W_0^{1,n}(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise}. \end{cases} \tag{8}$$

Furthermore, if f is positively homogeneous of degree n the function φ and the functional F_0 are independent of the subsequence, so that the whole family (F_δ) Γ-converges. In this case a simplified formula for φ holds.

The proof of this results relies on a general argument by Ansini and Braides [2], which reduces the computation of the 'extra term' along a sequence $u_\delta \to u$ to an estimate close to the perforation; i.e., on balls $B_{\rho\delta}(\delta i)$ for some small $\rho > 0$ (*a posteriori* independent of ρ). It is easily seen that the limit is not trivial only when $\varepsilon \ll \delta$ so that $K \subset B_{\rho\delta/\varepsilon}$ for ε small enough. If u is continuous and f is p-homogeneous this estimate reads:

$$\int_{B_{\rho\delta}(\delta i)} f(Du_\delta)\,\mathrm{d}x \geqslant \varepsilon^{n-p}\left|u(\delta i)\right|^p \min\left\{ \int_{B_{\rho\delta/\varepsilon}(0)} f(Dv)\,\mathrm{d}y\colon\ v = 0 \text{ on } K,\ v = 1 \text{ on } \partial B_{\rho\delta/\varepsilon}(0) \right\}. \tag{9}$$

When $p < n$ the minimum problem in (9) is estimated in [2] by the *p-capacity* of the set K (with respect to $\mathbb{R}^n$). Summing up in i, we obtain a Riemann sum provided that $\varepsilon^{n-p} = M\delta^n + o(\delta)$, which gives the scaling $\varepsilon = R\delta^{\frac{n}{n-p}}$. In the case $p = n$ the same argument gives a trivial lower bound since the corresponding limit computation of the n-capacity of K (with respect to $\mathbb{R}^n$) gives,

$$\inf\left\{ \int_{\mathbb{R}^n} |Dv|^n\,\mathrm{d}y\colon\ v = 0 \text{ on } K,\ 1 - v \in W^{1,n}(\mathbb{R}^n) \right\} = 0,$$

from which we deduce that the limit of the right-hand side of (9) is 0. We have therefore to depart from the proof in [2] by a more difficult analysis of the behaviour of the energies defined by the minimum problems in (9). This can be done explicitly if K is a ball, and gives (6) as a result. Note that in this case the radius of K does not affect the result; we can therefore extend the result to arbitrary K with nonempty interior by comparison with the case of balls containing K or contained in K, respectively, and conclude that the form of the limit is indeed independent of the shape of K. Further technical arguments are needed when f is not positively homogeneous; a detailed proof can be found in [9].

Remark 1. Using the terminology introduced in [5] our result can be summarized by saying that the functionals F_δ in (4) are equivalent to G_δ defined as

$$G_\delta(u) = \int_\Omega f(Du)\,\mathrm{d}x + \frac{|\log\varepsilon|^{n-1}}{\delta^n} \int_\Omega \varphi(u)\,\mathrm{d}x, \quad u \in W_0^{1,n}(\Omega;\mathbb{R}^m), \tag{10}$$

for $\delta \to 0$, meaning that both families have the same Γ-limits on all Γ-converging sequences with δ_j and ε_j tending to 0. Our arguments show that the exponential regime derives from the scaling invariance of the problems in (9), which eliminates the pre-factor ε^{n-p}, and from the logarithmic behaviour of minimizers. We have shown that the usual 'capacitary' formula for the limit integrand φ in the case $p < n$ is substituted by an interesting 'homogenization' formula. This highlights that in this critical case the energy does not concentrate at the same scale as the perforation radius, in a fashion similar to optimal sequences for Ginzburg–Landau functionals [3,8,1].

3. Asymptotic behaviour close to the critical scaling

Theorem 2.1, together with the companion analysis for $p < n$, shows a passage from a polynomial to an exponential decay of the relevant perforations at the critical scaling. To overcome the discontinuity in the description of the asymptotic analysis of energies (4) at $p = n$ we consider their dependence also on varying p. Since we are interested in a scale analysis, it is sufficient to consider the (scalar) case $f(Du) = |Du|^p$ and $K = \overline{B}_1$. We set:

$$F_\delta^p(u) = \begin{cases} \int_\Omega |Du|^p \,\mathrm{d}x & \text{if } u \in W_0^{1,p}(\Omega) \text{ and } u = 0 \text{ on } \Omega \setminus \Omega_\delta, \\ +\infty & \text{otherwise.} \end{cases} \tag{11}$$

By letting at the same time $\delta \to 0$ and $p \to n$ we can highlight three different behaviours of the perforation scaling. If $p - n \gg \delta^{n/n-1}$ then the functionals behave as in the case $p > n$ where every perforation gives a trivial limit since it enforces the constraint $u = 0$ on limits of sequences bounded in energy. In the other two regimes there exists a scaling giving a nontrivial limit. If $|p - n| = \mathrm{O}(\delta^{n/n-1})$ then the critical perforation scale is exponential as for $p = n$, while in the remaining case $n - p \gg \delta^{n/n-1}$ it is an interpolation between the exponential and the polynomial scaling. The precise form of the Γ-limit in dependence of the perforation is described by the following theorem, in which we also explicitly link the radii of the perforation to the coefficient κ of the additional term in the limit.

Theorem 3.1 *(Asymptotic behaviour close to the critical exponent). The Γ-limit of the energies F_δ^p defined in* (11) *as $\delta \to 0$ and $p = p(\delta) \to n$ exists and is described explicitly in the following three regimes. In the first two there exists a choice of the perforation $\varepsilon = \varepsilon(\delta)$ such that the limit is*:

$$F_0(u) = \begin{cases} \int_\Omega |Du|^n \,\mathrm{d}x + \kappa \int_\Omega |u|^n \,\mathrm{d}x & \text{if } u \in W_0^{1,n}(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{12}$$

The link between ε and κ is expressed by

(i) (Interpolation between the polynomial and the exponential regime) *if $p < n$ and $n - p \gg \delta^{n/n-1}$, then*

$$\varepsilon = R^{\frac{1}{n-p}} \delta^{\frac{n}{n-p}} (n-p)^{-\frac{(n-1)}{(n-p)}}, \qquad \kappa = R \frac{\omega_{n-1}}{(n-1)^{(n-1)}}, \quad \textit{with } R > 0;$$

(ii) (Exponential regime) *if $n - p = \gamma \delta^{n/n-1} + \mathrm{o}(\delta^{n/n-1})$ for $\gamma \in \mathbb{R}$ then*

$$\varepsilon = \exp(-a/\delta^{n/n-1}), \quad \kappa = \frac{\omega_{n-1}}{(n-1)^{(n-1)}} e^{-a\gamma} \left(\frac{1 - e^{-a\gamma/(n-1)}}{\gamma} \right)^{1-n}, \quad \textit{with } a > 0, \ \textit{if } \gamma \neq 0$$

$$\kappa = \frac{\omega_{n-1}}{a^{n-1}}, \quad \textit{with } a > 0, \ \textit{if } \gamma = 0.$$

Note that in this regime $n - p$ can also be negative;

(iii) (Rigid regime) *if $p > n$ and $p - n \gg \delta^{n/n-1}$ then the limit is finite* (*and null*) *only on the constant function zero* (*this can be seen as a degenerate case on* (12) *when we take $\kappa = +\infty$*).

The proof of this result relies on adapting the arguments in [2] to the case of varying exponent p, to carefully estimate the minimum problems in (9), and understand their interplay with the (unknown) prefactor ε^{n-p}. A detailed proof will appear in [10].

Remark 2. Our arguments show that the functionals F_δ^p are equivalent to G_δ^p given by:

$$G_\delta^p(u) = \int_\Omega |Du|^p \,\mathrm{d}x + C_p \frac{\varepsilon^{n-p}}{\delta^n} \left(\frac{1 - \varepsilon^{n-p/p-1}}{n-p} \right)^{1-p} \int_\Omega |u|^p \,\mathrm{d}x, \quad u \in W_0^{1,p}(\Omega; \mathbb{R}^m) \tag{13}$$

(C_p explicitly computable) as $p \to n$. This description can be extended to all $p > 1$ upon noticing that the scaling in regime (i) reduces to the usual polynomial perforation scaling for $p < n$ fixed. It is worth noting that this description is uniform in p, in the sense that the Γ-limits of Γ-converging subsequences of the two families of energies are the same also if we let p vary as δ and ε tend to 0. A general analysis of this kind of uniform equivalence between functionals can be found in [5].

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