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Numerical Analysis

A posteriori error estimation in the conforming finite element method based on its local conservativity and using local minimization ☆

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Abstract

We present in this Note fully computable a posteriori error estimates allowing for accurate error control in the conforming finite element discretization of pure diffusion problems. The derived estimates are based on the local conservativity of the conforming finite element method on a dual grid associated with simplex vertices rather than directly on the Galerkin orthogonality. ***To cite this article: M. Vohralík, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Estimation d'erreur a posteriori dans la méthode des éléments finis conformes basée sur sa conservativité locale et employant une minimisation locale. Nous présentons dans cette Note des estimations a posteriori entièrement calculables, permettant le contrôle d'erreur dans la discrétisation de problèmes à diffusion pure par la méthode des éléments finis conformes. Ces estimations sont basées sur la conservativité locale de la méthode des éléments finis conformes sur un maillage dual associé aux sommets des triangles ou tétraèdres au lieu de l'orthogonalité de Galerkine. ***Pour citer cet article : M. Vohralík, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

A posteriori error estimates for the conforming finite element approximation of the model problem

$$-\Delta p = f \quad \text{in } \Omega, \tag{1a}$$

$$p = 0 \quad \text{on } \partial\Omega, \tag{1b}$$

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where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal (polyhedral) domain, have been subject to a vast literature in the last decades. Several branches of estimators have been developed, such as the averaging, residual, or equilibrated residual ones. The majority of these estimators have been proved both reliable (yielding a global upper bound) and locally efficient (giving a local lower bound), but not guaranteed in the sense that the upper bound does not contain any unknown constant. We have recently in [9,8,5] introduced estimators for mixed finite element, finite volume, and discontinuous Galerkin methods which are fully computable and give tight upper bounds. The purpose of this note is to present their extension to the conforming finite element method, completing thus the few existing guaranteed estimates in this case, cf., e.g., Destuynder and Métivet [4], Luce and Wohlmuth [6], or Repin and Sauter [7].

We first in Section 2 give an optimal abstract framework for the estimation of the energy error between the weak solution of (1a)–(1b) and an arbitrary conforming function. Using this framework, we then in Section 3 derive our a posteriori error estimates. Finally, in Section 4 we outline how the presented estimators can still be improved using local minimization. Numerical experiments, details on the local minimization, and comparisons with the standard residual-based estimators are given in [2]. Detailed proofs, extensions to other methods giving approximations in conforming spaces (cell-centered finite volume, vertex-centered finite volume, and finite difference ones), and extension to the problem $-\nabla \cdot (a \nabla p) = f$ in Ω, where a is a scalar, piecewise constant, and arbitrarily discontinuous diffusion coefficient and $f \in L^2(\Omega)$, are then given in [10]. In particular, the estimates presented in this reference are fully robust with respect to the diffusion coefficient a without any condition on the monotonicity of its distribution. Finally, an extension to the reaction–diffusion case, leading again to guaranteed and robust a posteriori error estimates, is given in [3].

2. Optimal abstract framework for a posteriori error estimation

Let us define a bilinear form $\mathcal{B}$ by $\mathcal{B}(p, \varphi) := (\nabla p, \nabla \varphi)_\Omega$, $p, \varphi \in H_0^1(\Omega)$ and the corresponding energy norm by

$$|||\varphi|||^2 := \mathcal{B}(\varphi, \varphi). \tag{2}$$

The weak formulation of problem (1a)–(1b) is then to find $p \in H_0^1(\Omega)$ such that

$$\mathcal{B}(p, \varphi) = (f, \varphi)_\Omega \quad \forall \varphi \in H_0^1(\Omega). \tag{3}$$

The following simple theorem gives an optimal abstract a posteriori error estimate:

Theorem 2.1 *(Abstract a posteriori error estimate and its efficiency). Let p be the weak solution of problem* (1a)–(1b) *given by* (3) *and let $p_h \in H_0^1(\Omega)$ be arbitrary. Then*

$$|||p - p_h||| = \inf_{\mathbf{t} \in \mathbf{H}(\mathrm{div}, \Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \big\{ (f - \nabla \cdot \mathbf{t}, \varphi)_\Omega - (\nabla p_h + \mathbf{t}, \nabla \varphi)_\Omega \big\}. \tag{4}$$

Proof. We first notice that $|||p - p_h||| = \mathcal{B}(p - p_h, p - p_h)/|||p - p_h|||$ by (2). Clearly, as $\varphi := (p - p_h)/|||p - p_h||| \in H_0^1(\Omega)$, we immediately have $\mathcal{B}(p, \varphi) = (f, \varphi)_\Omega$ by (3). Using this we obtain, for an arbitrary $\mathbf{t} \in \mathbf{H}(\mathrm{div}, \Omega)$ and employing the Green theorem,

$$\begin{aligned} \mathcal{B}(p - p_h, \varphi) &= (f, \varphi)_\Omega - (\nabla p_h, \nabla \varphi)_\Omega = (f, \varphi)_\Omega - (\nabla p_h + \mathbf{t}, \nabla \varphi)_\Omega + (\mathbf{t}, \nabla \varphi)_\Omega \\ &= (f - \nabla \cdot \mathbf{t}, \varphi)_\Omega - (\nabla p_h + \mathbf{t}, \nabla \varphi)_\Omega, \end{aligned}$$

whence it immediately follows that (4) holds with the $\leqslant$ sign. For the converse, it suffices to put $\mathbf{t} = -\nabla p$ and to use the Schwarz inequality and the fact that $|||\varphi||| = 1$. □

3. Guaranteed and fully computable a posteriori error estimates for the conforming finite element method based on its local conservativity

Let now $\mathcal{T}_h$ be a conforming simplicial mesh of Ω, $\mathcal{V}_h$ ($\mathcal{V}_h^{\mathrm{int}}$, $\mathcal{V}_h^{\mathrm{ext}}$) the set of all (interior, exterior) vertices of $\mathcal{T}_h$, and $\mathcal{T}_V := \{L \in \mathcal{T}_h;\ L \cap V \neq \emptyset\}$ for $V \in \mathcal{V}_h$. We will use the discrete space given as $X_h^0 := \{\varphi_h \in H_0^1(\Omega);\ \varphi_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h\}$, where $\mathbb{P}_1(K)$ denotes the space of linear polynomials on K. The basis of this space is spanned by the classical pyramidal functions ψ_V, $V \in \mathcal{V}_h^{\mathrm{int}}$, such that $\psi_V(U) = \delta_{VU}$, $U \in \mathcal{V}_h$. The conforming finite element method for the problem (1a)–(1b) then consists in finding $p_h \in X_h^0$ such that

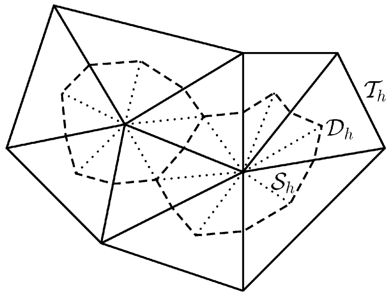

Fig. 1. Original simplicial mesh $\mathcal{T}_h$, the associated dual mesh $\mathcal{D}_h$, and the fine simplicial mesh $\mathcal{S}_h$.

$$\mathcal{B}(p_h, \varphi_h) = (f, \varphi_h)_\Omega \quad \forall \varphi_h \in X_h^0. \tag{5}$$

In addition to $\mathcal{T}_h$, we shall here consider also dual partitions $\mathcal{D}_h$ of Ω. A dual volume D associated with vertex V_D is constructed by connecting the barycentres of simplices from $\mathcal{T}_{V_D}$ through edge midpoints (and face barycentres if $d = 3$), see Fig. 1. The obvious notations $\mathcal{D}_h^{\mathrm{int}}$, $\mathcal{D}_h^{\mathrm{ext}}$ are used and h_D stands for the diameter of $D \in \mathcal{D}_h$. Finally, we will need a second simplicial mesh $\mathcal{S}_h$ of Ω, constructed by dividing each $D \in \mathcal{D}_h$ into a mesh $\mathcal{S}_D$ as indicated in Fig. 1. We will use the notation $\mathcal{G}_h$ for all sides of $\mathcal{S}_h$ and $\mathcal{G}_h^{\mathrm{int}}$ ($\mathcal{G}_h^{\mathrm{ext}}$) for all interior (exterior) sides of $\mathcal{S}_h$. Next, $\mathbf{n}$ denotes an exterior normal vector, whereas $\mathbf{n}_\sigma$ a normal vector of a side σ, arbitrary but fixed. Finally, the average operator $\{\{\cdot\}\}$ is defined by $\{\{\varphi\}\} := \frac{1}{2}(\varphi|_K)|_\sigma + \frac{1}{2}(\varphi|_L)|_\sigma$ for $\sigma \in \mathcal{G}_h^{\mathrm{int}}$ shared by $K, L \in \mathcal{S}_h$; for $\sigma \in \mathcal{G}_h^{\mathrm{ext}}$, $\{\{\varphi\}\} := \varphi|_\sigma$.

In order to give our a posteriori error estimate, we now construct a particular $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$. It will be defined in the lowest-order Raviart–Thomas–Nédélec space **RTN** over the fine simplicial mesh $\mathcal{S}_h$; recall that the constant values $\mathbf{v} \cdot \mathbf{n}_\sigma$ represent the degrees of freedom of $\mathbf{RTN}(\mathcal{S}_h)$. We define $\mathbf{t}_h \in \mathbf{RTN}(\mathcal{S}_h)$ by

$$\mathbf{t}_h \cdot \mathbf{n}_\sigma = -\{\{\nabla p_h \cdot \mathbf{n}_\sigma\}\} \quad \forall \sigma \in \mathcal{G}_h. \tag{6}$$

Note that $\mathbf{t}_h \cdot \mathbf{n}_\sigma$ are given directly by $-\nabla p_h \cdot \mathbf{n}_\sigma$ for all the sides $\sigma \in \mathcal{G}_h$ which are in the interior of some $K \in \mathcal{T}_h$ or at the boundary of Ω, whereas a simple average of the two normal gradient values is used otherwise. In terms of $\mathbf{t}_h$, the following local conservation property holds:

Lemma 3.1 *(Local conservativity of the conforming finite element method on the dual grid). Let f be piecewise constant on $\mathcal{T}_h$ and let $\mathbf{t}_h$ by given by* (6). *Then*

$$(\nabla \cdot \mathbf{t}_h, 1)_D = (f, 1)_D \quad \forall D \in \mathcal{D}_h^{\mathrm{int}}. \tag{7}$$

Proof. Employing the Green theorem, the finite elements basis functions form, and the relation between $\mathcal{T}_h$ and $\mathcal{D}_h$, see [1, Lemma 3] for $d = 2$, it is straightforward to prove that $(\nabla p_h, \nabla \psi_{V_D})_{\mathcal{T}_{V_D}} = -\langle \nabla p_h \cdot \mathbf{n}, 1\rangle_{\partial D}$ $\forall D \in \mathcal{D}_h^{\mathrm{int}}$. Next, under the assumption on f and using that $|D \cap K| = |K|/(d+1)$ for $D \in \mathcal{D}_h^{\mathrm{int}}$ and $K \in \mathcal{T}_{V_D}$, $(f, \psi_{V_D})_{\mathcal{T}_{V_D}} = (f, 1)_D$ $\forall D \in \mathcal{D}_h^{\mathrm{int}}$. Thus (5) implies $-\langle \nabla p_h \cdot \mathbf{n}, 1\rangle_{\partial D} = (f, 1)_D$ for all $D \in \mathcal{D}_h^{\mathrm{int}}$, whence $\langle \mathbf{t}_h \cdot \mathbf{n}, 1\rangle_{\partial D} = (f, 1)_D$ for all $D \in \mathcal{D}_h^{\mathrm{int}}$, using the definition (6) and the fact that all the sides of $D \in \mathcal{D}_h^{\mathrm{int}}$ lie in the interior of some $K \in \mathcal{T}_h$. The assertion of the lemma now follows by the Green theorem. □

Before stating our estimate, let us recall two basic inequalities. For $D \in \mathcal{D}_h^{\mathrm{int}}$, the Poincaré inequality states that

$$\|\varphi - \varphi_D\|_D^2 \leqslant C_{\mathrm{P},D} h_D^2 \|\nabla \varphi\|_D^2 \quad \forall \varphi \in H^1(D), \tag{8}$$

where $\varphi_D := (\varphi, 1)_D/|D|$. For convex D, $C_{\mathrm{P},D} = 1/\pi^2$. For $D \in \mathcal{D}_h^{\mathrm{ext}}$, the Friedrichs inequality states that

$$\|\varphi\|_D^2 \leqslant C_{\mathrm{F},D,\partial\Omega} h_D^2 \|\nabla \varphi\|_D^2 \quad \forall \varphi \in H^1(D) \text{ such that } \varphi = 0 \text{ on } \partial\Omega \cap \partial D \tag{9}$$

and under fairly general conditions, $C_{\mathrm{F},D,\partial\Omega} = 1$. We refer to [10] for more details.

Theorem 3.2 *(Guaranteed and fully computable a posteriori error estimate). Let f be piecewise constant on $\mathcal{T}_h$, let p be the weak solution of problem* (1a)–(1b) *given by* (3)*, and let p_h be its conforming finite element approximation given by* (5)*. Let next $\mathbf{t}_h$ be given by* (6)*. Then*

$$|||p - p_h||| \leqslant \left\{ \sum_{D \in \mathcal{D}_h} (\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D})^2 \right\}^{1/2},$$

where the diffusive flux estimator $\eta_{\mathrm{DF},D}$ *is given by* $\eta_{\mathrm{DF},D} := \|\nabla p_h + \mathbf{t}_h\|_D$, $D \in \mathcal{D}_h$, *and the* residual estimator $\eta_{\mathrm{R},D}$ *is given by* $\eta_{\mathrm{R},D} := m_D \|f - \nabla \cdot \mathbf{t}_h\|_D$, $D \in \mathcal{D}_h$, *where* $m_D^2 := C_{\mathrm{P},D} h_D^2$, *if* $D \in \mathcal{D}_h^{\mathrm{int}}$, *and* $m_D^2 := C_{\mathrm{F},D,\partial\Omega} h_D^2$, *if* $D \in \mathcal{D}_h^{\mathrm{ext}}$.

Proof. Put $\mathbf{t} = \mathbf{t}_h$ in Theorem 2.1. Note that, for each $D \in \mathcal{D}_h^{\mathrm{int}}$, $(f - \nabla \cdot \mathbf{t}_h, \varphi)_D = (f - \nabla \cdot \mathbf{t}_h, \varphi - \varphi_D)_D \leqslant \eta_{\mathrm{R},D} |||\varphi|||_D$, using Lemma 3.1, the Schwarz inequality, the Poincaré inequality (8), and (2). For $D \in \mathcal{D}_h^{\mathrm{ext}}$, likewise, $(f - \nabla \cdot \mathbf{t}_h, \varphi)_D \leqslant \eta_{\mathrm{R},D} |||\varphi|||_D$, using the Schwarz inequality, the Friedrichs inequality (9), and (2). Finally, using the Schwarz inequality and (2), $-(\nabla p_h + \mathbf{t}_h, \nabla\varphi)_D \leqslant \eta_{\mathrm{DF},D} |||\varphi|||_D$ is immediate. Hence it now suffices to use the Cauchy–Schwarz inequality and to notice that $|||\varphi||| = 1$ in order to conclude the proof. □

Finally, the proof of the following theorem can be found in [10]:

Theorem 3.3 *(Local efficiency of the a posteriori error estimate). Let the assumptions of Theorem* 3.2 *be verified and let in addition* $\mathcal{T}_h$ *be shape-regular. Then, for each* $D \in \mathcal{D}_h$, *there holds*

$$\eta_{\mathrm{DF},D} \leqslant C |||p - p_h|||_D, \qquad \eta_{\mathrm{R},D} \leqslant \tilde{C} |||p - p_h|||_D,$$

where the constant C *depends only on the space dimension* d *and on the shape regularity of the mesh and the constant* $\tilde{C}$ *in addition depends on* $C_{\mathrm{P},D}$ *from the Poincaré inequality* (8) *if* $D \in \mathcal{D}_h^{\mathrm{int}}$ *or on the constant* $C_{\mathrm{F},D,\partial\Omega}$ *from the Friedrichs inequality* (9) *if* $D \in \mathcal{D}_h^{\mathrm{ext}}$.

4. Improvements using local minimization

The numerical experiments presented in [2,10] confirm the above theoretical results but show that the effectivity index (ratio of the estimated and actual error) does not approach the optimal value of one as it is the case in [9,8]. It turns out that whereas in these references, the residual estimator represents a higher-order term, it is not the case here, as (7) is only true on a set of elements $\mathcal{S}_D$ of $D \in \mathcal{D}_h^{\mathrm{int}}$ and not on each element $K \in \mathcal{S}_D$. However, the estimate of Theorem 3.2 obviously holds true for any $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ such that (7) is verified, and in particular for any $\mathbf{t}_h \in \mathbf{RTN}(\mathcal{S}_h)$ such that (6) only holds for such $\sigma \in \mathcal{G}_h$ which are at the boundary of some $D \in \mathcal{D}_h^{\mathrm{int}}$.

We thus in [2] investigate an approach where $\mathbf{t}_h \cdot \mathbf{n}_\sigma$ for $\sigma \in \mathcal{G}_h$ which are not at the boundary of some $D \in \mathcal{D}_h^{\mathrm{int}}$ are given by local minimization of $\eta_{\mathrm{R},D}^2 + \eta_{\mathrm{DF},D}^2$ for each $D \in \mathcal{D}_h$. This leads to a solution of a small linear system for each $D \in \mathcal{D}_h$ and helps the improve the effectivity index to a value close to one. Another approach, avoiding any linear system solution, is pursued in [10]. Here $\mathbf{t}_h$ used in the estimate of Theorem 3.2 is given by $\alpha\mathbf{t}_{1,h} + (1-\alpha)\mathbf{t}_{2,h}$, with $\mathbf{t}_{1,h}$ given by (6) and $\mathbf{t}_{2,h}$ given by (6) only for such $\sigma \in \mathcal{G}_h$ which are at the boundary of some $D \in \mathcal{D}_h^{\mathrm{int}}$ and such that $(\nabla \cdot \mathbf{t}_{2,h}, 1)_K = (f, 1)_K$ for all $K \in \mathcal{S}_D$ and all $D \in \mathcal{D}_h$.

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