Available online at www.sciencedirect.com

ScienceDirect

C. R. Acad. Sci. Paris, Ser. I 346 (2008) 487–489

COMPTES RENDUS MATHEMATIQUE

http://france.elsevier.com/direct/CRASS1/

Number Theory

On binary palindromes of the form $10^n \pm 1$ ☆

Florian Luca [a], Alain Togbé [b]

[a] *Instituto de Matemáticas UNAM, Campus Morelia Apartado Postal 27-3 (Xangari), C.P. 58089, Morelia, Michoacán, Mexico*
[b] *Mathematics Department, Purdue University North Central, 1401 S, U.S. 421, Westville, IN 46391, USA*

Received 30 August 2007; accepted after revision 11 March 2008

Available online 14 April 2008

Presented by Christophe Soulé

Abstract

In this Note, we find all positive integers n such that $10^n \pm 1$ is a binary palindome. Our proof uses lower bounds for linear forms in logarithms of rational numbers. ***To cite this article: F. Luca, A. Togbé, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Sur des palindromes binaires du format $10^n \pm 1$. Dans cette Note, nous trouvons tous les entiers positifs n tels que $10^n \pm 1$ soit un palindrome binaire. Notre démontration utilise les minorations de formes linéaires en logarithmes de nombres rationnels. ***Pour citer cet article : F. Luca, A. Togbé, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

Let $b \geqslant 1$ be an integer. A *base b palindrome* is a positive integer whose string of digits when written in base b reads the same from left to right as from right to left. That is, if the base b representation of n is $n = \overline{a_1 a_2 \cdots a_m}_{(b)}$, then $a_i = a_{m-i+1}$ holds for all $i = 1, \ldots, m$. For example, 123 454 321 is a base 10 palindrome, and $7 = \overline{111}_{(2)}$ is a binary palindrome. Palindromes are fascinating numbers to mathematicians and aficionados alike who find a lot of joy when accidentally stumbling upon a palindrome in real life. The construction of the famous Charles Bridge over the Vltava river was started by King Charles IV at 5:31am on the 9th of July, 1357 this minute being then enumerated as 135 797 531. It seems to be very difficult to prove any strong results about the presence of palindromes in various arithmetically interesting sequences. For example, it is not known if there are infinitely many prime palindromes although this is conjectured to be so. In fact, it is not even known if there are infinitely many square-free palindromes but it is known that there are infinitely many cube free ones! Several interesting results about prime factors of palindromes can be found in [2] and [3]. Some recreational results about palindromic squares can be found in [1] and [4].

☆ Work by the first author was done in the Summer of 2007 when he visited the School of Mathematics of the Tata Institute in Mumbai, India. He thanks the host institution for its hospitality and the Third World Academy of Sciences for support. The second author was partially supported by Purdue University North Central.

E-mail addresses: fluca@matmor.unam.mx (F. Luca), atogbe@pnc.edu (A. Togbé).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2008.03.015

Given any positive integer n, both $10^n+1=\overline{100\cdots01}_{(10)}$ (with $n-1$ zeros), as well as $10^n-1=\overline{99\cdots9}_{(10)}$ (with n nines) are palindromes in base 10. In this Note, we look at the values of n such that $10^n\pm1$ is a base 2 palindrome.

In [6], it was shown that the set of positive integers n such that F_n, the nth Fibonacci number, is a base b palindrome is of asymptotic density zero. Almost certainly this set is finite for any given b, but this has not been proved yet. The result proved there in fact applies with the Fibonacci sequence $(F_n)_{n\geqslant0}$ replaced by any Lucas sequence whose characteristic roots are quadratic units. In the last section of [6] it is shown how one may adapt the arguments used there to deal also with sequences of the form $a^n\pm1$, where a and b multiplicatively independent. The condition that a and b are multiplicatively independent is needed for otherwise one can easily show that the set of n such that $a^n\pm1$ is a base b palindrome contains an infinite arithmetical progression. When additionally all the prime factors of b divide a, it is pointed out, without proof, that the set of such n is finite and effectively computable. In this note, we give the details of the proof of this assertion and perform the computations for the case when $a=10$ and $b=2$.

Our result is the following:

Theorem 1.1. *The only positive integers n such that $10^n\pm1$ are binary palindromes are $1,2$, i.e. $9=\overline{1001}_{(2)}$ and $99=\overline{1100011}_{(2)}$.*

2. The proof

We assume first that $n>1000$. We put m for the number of binary digits of $N=10^n\pm1$. Since none of 10^n and $10^n\pm1$ is a power of 2, it follows that

$$m=\big\lfloor\big(\log(10^n\pm1)\big)/(\log2)\big\rfloor+1=\big\lfloor n(\log10)/(\log2)\big\rfloor+1=n+\big\lfloor n(\log5)/(\log2)\big\rfloor+1.$$

Put $\ell=\lfloor n(\log5)/(\log2)\rfloor+1$ for the number of binary digits of 5^n. The last binary $n+1$ digits of N are $100\ldots01$ with $n-1$ zeros, or $011\ldots11$ with n ones, according to whether the sign is $+$ or $-$, respectively. Since N is a binary palindrome, the first binary digits of N must be $100\ldots01$ with $n-1$ zeros and $11\ldots110$ with n ones, when the sign is $+$ and respectively $-$. Since $2<5$, the first n binary digits of N are the same as the first n binary digits of 5^n. In fact, this is true for the first $\ell-1\geqslant n$ binary digits of N. Thus, in the $+$ case, we have

$$2^\ell<5^n\leqslant2^\ell+2^{\ell-n}+2^{\ell-n-1}+\cdots+1<2^\ell+2^{\ell-n+1},$$

while in the $-$ case

$$2^{\ell+1}>5^n>2^\ell+2^{\ell-1}+\cdots+2^{\ell-n+1}=2^{\ell-n+1}(2^{n-1}+\cdots+1)=2^{\ell-n+1}(2^n-1)=2^{\ell+1}-2^{\ell-n+1}.$$

Putting $k=\ell$ or $\ell+1$ according to whether the sign is $+$ or $-$, we get that

$$|5^n2^{-k}-1|<2^{\ell-n+1-k}\leqslant2^{-n+1}.$$

Put $\Lambda=n\log5-k\log2$. The above inequality becomes $|e^\Lambda-1|<2^{-n+1}$. If $\Lambda>0$, then

$$|\Lambda|=\Lambda<e^\Lambda-1<2^{-n+1},$$

while if $\Lambda<0$, then the inequality

$$|e^\Lambda-1|=1-e^{-|\Lambda|}<2^{-n+1}$$

leads to

$$|\Lambda|<e^{|\Lambda|}-1<\frac{1}{1-2^{-n+1}}-1=\frac{2^{-n+1}}{1-2^{-n+1}}<2^{-n+2}$$

because $n>2$ (thus, $2^{-n+1}<1/2$). In conclusion, the inequality

$$|\Lambda|<2^{-n+2}$$

holds in both cases and it implies

$$(n-2)\log2<-\log|\Lambda|. \tag{1}$$

We have now set ourselves up to apply a lower bound for a linear form in logarithms. There are many such in the literature. We have chosen to apply the following one which is Corollary 2 on page 228 in [5]. First, let us introduce some notations. Let a_1 and a_2 be integers > 1, b_1 and b_2 be positive integers, A_1 and A_2 be positive real numbers such that $A_i \geqslant \max\{a_i, e\}$ for $i = 1, 2$, and $\Lambda = b_2 \log a_2 - b_1 \log a_1$. Let

$$b' = \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1}.$$

With these notations, the above mentioned result from [5] asserts that

$$-\log|\Lambda| \leqslant 24.34\bigl(\max\{\log b' + 0.14, 21\}\bigr)^2 \log A_1 \log A_2. \tag{2}$$

For us, we can take $a_1 = 2$, $a_2 = 5$, $b_1 = k$, $b_2 = n$, $A_1 = e$ and $A_2 = 5$. Thus,

$$\begin{aligned} b' &= \frac{k}{\log 5} + n \leqslant \frac{\ell + 1}{\log 5} + n \leqslant \frac{1}{\log 5}\left(n\frac{\log 5}{\log 2} + 2\right) + n \leqslant n + \frac{n}{\log 2} + \frac{2}{\log 5} \\ &< 3n + 2. \end{aligned} \tag{3}$$

Combining estimates (1) with (2) and (3), we get

$$(n-2)\log 2 < 24.34 \max\bigl\{\log(3n+2) + 0.14, 21\bigr\}^2 \log 5. \tag{4}$$

The last inequality above implies that $n \leqslant 5500$. A few seconds of computation with Mathematica revealed that in the range $n \leqslant 5500$, there are no other binary palindromes among $10^n \pm 1$ except for the two shown in the statement of the theorem, which completes the proof.

Acknowledgement

We thank the referee for a careful reading of the manuscript and for useful suggestions.

References

[1] C. Ashbacher, More on palindromic squares, J. Recreational Math. 22 (1990) 133–135.
[2] W.D. Banks, D.N. Hart, M. Sakata, Almost all palindromes are composite, Math. Res. Lett. 11 (2004) 853–868.
[3] W.D. Banks, I. Shparlinski, Prime divisors of palindromes, Period. Math. Hungar. 51 (2005) 1–10.
[4] M. Keith, Classification and enumeration of palindromic squares, J. Recreational Math. 22 (1990) 124–132.
[5] M. Laurent, M. Mignotte, Yu. Nesterenko, Formes linéaires en deux logarithmes et déterminants d'interpolation, J. Number Theory 55 (1995) 285–321.
[6] F. Luca, Palindromes in Lucas sequences, Monatsh. Math. 138 (2003) 209–223.