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Differential Geometry

A uniform Sobolev inequality for Ricci flow with surgeries and applications

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Abstract

We prove a uniform Sobolev inequality for Ricci flow, which is independent of the number of surgeries. As an application, under less assumptions, a noncollapsing result stronger than Perelman's κ noncollapsing with surgery is derived. The proof is much shorter and seems more accessible. The result also improves some earlier ones where the Sobolev inequality depended on the number of surgeries. ***To cite this article: Q.S. Zhang, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Une inégalité de Sobolev uniforme pour le flot de Ricci avec chirurgie et applications. Nous prouvons une inégalité de Sobolev uniforme pour le flot de Ricci, indépendante du nombre de chirurgies. Comme application, nous établissons, avec moins d'hypothèses, un résultat de non-explosion plus fort que celui de Perelman sur la non-explosion de κ avec chirurgie. La preuve est plus courte et semble plus accessible. Le résultat améliore également des résultats antérieurs où l'inégalité de Sobolev dépendait du nombre de chirurgies. ***Pour citer cet article : Q.S. Zhang, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction and statements of results

A crucial step in Perelman's work on Poincaré and Geometrization conjectures is the κ noncollapsing result for Ricci flow with or without surgeries. The proof of this result in the surgery case requires truly complicated calculation using such new concepts as reduced distance, admissible curve, barely admissible curve, gradient estimate of scalar curvature etc. This is elucidated in great length by Cao and Zhu [3], Kleiner and Lott [2] and Morgan and Tian [7].

In this Note we prove a uniform Sobolev inequality for Ricci flow, which is independent of the number of surgeries. It is well known that uniform Sobolev inequalities are essential in that they encode rich analytical and geometrical information on the manifold. These include, noncollapsing, isoperimetric inequalities etc. As a consequence, a strong noncollapsing result is obtained. It includes Perelman's κ noncollapsing with surgery as a special case. The result also requires less assumptions. For instance we do not need the canonical neighborhood assumption for the whole manifold

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(see Remark 1.2 below). In the proof, we use only Perelman's W entropy and some analysis of the minimizer equation of the W entropy on horn like manifolds. Hence it is shorter and seems more accessible.

Let $\mathbf{M}$ be a compact Riemannian manifold of dimension $n \geqslant 3$ and g be the metric. Then a Sobolev inequality of the following form holds: there exist positive constants A, B such that, for all $v \in W^{1,2}(\mathbf{M}, g)$,

$$\left(\int v^{2n/(n-2)}\,\mathrm{d}\mu(g)\right)^{(n-2)/n} \leqslant A\int |\nabla v|^2\,\mathrm{d}\mu(g) + B\int v^2\,\mathrm{d}\mu(g). \tag{1}$$

This inequality was proven by Aubin [1] for $A = K^2(n) + \epsilon$ with $\epsilon > 0$ and B depending on bounds on the injectivity radius, sectional curvatures. Here $K(n)$ is the best constant in the Sobolev embedding for $\mathbf{R}^n$. Hebey [4] showed that B can be chosen to depend only on ϵ, the injectivity radius and the lower bound of the Ricci curvature. Hebey and Vaugon [5] proved that one can even take $\epsilon = 0$. See also [6]. However the constant B will also depend on the derivatives of the curvature tensor. Hence, the controlling geometric quantities for B as stated above are not invariant under the Ricci flow in general. Theorem 1.1 below states that a uniform Sobolev inequality of the above type holds uniformly under Ricci flow in finite time, even in the presence of indefinite number of surgeries.

In order to state the theorem, we first introduce some notations. They are mainly taken from [8,9,3,2] and [7].

We use $(\mathbf{M}, g(t))$ to denote Hamilton's Ricci flow, $\frac{\mathrm{d}g}{\mathrm{d}t} = -2Ric$. If a surgery occurs at time t, then $(\mathbf{M}, g(t^-))$ denotes the pre surgery manifold (the one right before the surgery); and $(\mathbf{M}, g(t^+))$ denotes the post surgery manifold (the one right after the surgery). The ball of radius r with respect to the metric $g(t)$, centered at x, is denoted by $B(x, t, r)$. The scalar curvature is denoted by $R = R(x, t)$ and $R_0^- = \sup R^-(x, 0)$. Rm denotes the full curvature tensor. $\mathrm{d}\mu(g(t))$ denotes the volume element. $\mathrm{vol}(\mathbf{M}(\mathbf{g}(\mathbf{t}))$ is the total volume of $\mathbf{M}$ under $g(t)$.

In this paper we use the following definition of κ noncollapsing by Perelman [9], as elucidated in Definition 77.9 of [2]:

Definition 1.1 *(κ noncollapsing).* Let $(\mathbf{M}, g(t))$ be a Ricci flow with surgery defined on $[a, b]$. Suppose that $x_0 \in \mathbf{M}$, $t_0 \in [a, b]$ and $r > 0$ are such that $t_0 - r^2 \geqslant a$, $B(x_0, t_0, r) \subset \mathbf{M}$ is a proper ball and the parabolic ball $P(x_0, t_0, r, -r^2)$ is unscathed. Then $\mathbf{M}$ is κ-collapsed at (x_0, t_0) at scale r if $|Rm| \leqslant r^{-2}$ on $P(x_0, t_0, r, -r^2)$ and $\mathrm{vol}(B(x_0, t_0, r)) < \kappa r^3$; otherwise it is κ-noncollapsed.

Here we introduce

Definition 1.2 *(Strong κ noncollapsing).* Let $\mathbf{M}$ be a Ricci flow with surgery defined on $[a, b]$. Suppose that $x_0 \in \mathbf{M}$, $t_0 \in [a, b]$ and $r > 0$ are such that $B(x_0, t_0, r) \subset \mathbf{M}$ is a proper ball. Then $\mathbf{M}$ is strong κ-noncollapsed at (x_0, t_0) at scale r if $R \leqslant r^{-2}$ on $B(x_0, t_0, r)$ and $\mathrm{vol}(B(x_0, t_0, r)) \geqslant \kappa r^3$.

This strong κ noncollapsing improves the κ noncollapsing on two aspects. One is that only information on the metric balls on one time level is needed. Thus it bypasses the complicated issue that a parabolic ball may be cut by a surgery. The other is that it only requires scalar curvature upper bound.

Definition 1.3 *(Normalized manifold).* A compact Riemannian manifold is normalized if $|Rm| \leqslant 1$ everywhere and the volume of every unit ball is at least half of the volume of the Euclidean unit ball.

Definition 1.4 *(ϵ neck, ϵ horn, double ϵ horn, and ϵ tube).* An ϵ neck (of radius r) is an open set with a metric which is, after scaling the metric with factor r^{-2}, ϵ close, in the $C^{\epsilon^{-1}}$ topology, to the standard neck $S^2 \times (-\epsilon^{-1}, \epsilon^{-1})$. Here and later $C^{\epsilon^{-1}}$ means $C^{[\epsilon^{-1}]+1}$.

Let I be an open interval in R^1. An ϵ horn (of radius r) is $S^2 \times I$ with a metric with the following properties: each point is contained in some ϵ neck; one end is contained in an ϵ neck of radius r; the scalar curvature tends to infinity at the other end.

An ϵ tube is $S^2 \times I$ with a metric such that each point is contained in some ϵ neck and the scalar curvature stays bounded on both ends.

A double ϵ horn is $S^2 \times I$ with a metric such that each point is contained in some ϵ neck and the scalar curvature tends to infinity at both ends.

Definition 1.5. A standard capped infinite cylinder is $\mathbf{R}^3$ equipped with a rotationally symmetric metric with non-negative sectional curvature and positive scalar curvature such that outside a compact set it is a semi-infinite standard round cylinder $S^2 \times (-\infty, 0)$.

Here is the main result of paper:

Theorem 1.1. *Given real numbers $T_1 < T_2$, let $(\mathbf{M}, g(t))$ be a $n = 3$ dimensional Ricci flow with normalized initial condition defined on the time interval containing $[T_1, T_2]$. Suppose the following conditions are met.*

(a) *There are finitely many (r, δ) surgeries in $[T_1, T_2]$, occurring in ϵ horns of radii r. Here $r \leqslant r_0$ and $\epsilon \leqslant \epsilon_0$, with r_0 and ϵ_0 being fixed sufficiently small positive numbers less than 1. The surgery radii are $h \leqslant \delta^2 r$ i.e. the surgeries occur in δ necks of radius $h \leqslant \delta^2 r$. Here $0 < \delta \leqslant \delta_0$ where $\delta_0 = \delta_0(r_0, \epsilon_0) > 0$ is sufficiently small. Outside of the ϵ horns, the Ricci flow is smooth.*

(b) *For a constant $c > 0$ and any point x in all the above ϵ horns, the following holds: there is a region U, satisfying, $B(x, c\epsilon^{-1}R^{-1/2}(x)) \subset U \subset B(x, 2c\epsilon^{-1}R^{-1/2}(x))$, such that, after scaling by a factor $R(x)$, it is ϵ close in the $C^{\epsilon^{-1}}$ topology to $S^2 \times (-\epsilon^{-1}, \epsilon^{-1})$.*

Also for any x in the modified part of the ϵ horn immediately after a surgery, the following holds: the ball $B(x, \epsilon^{-1}R^{-1/2}(x))$, is, after scaling by a factor $R(x)$, ϵ close in the $C^{\epsilon^{-1}}$ topology to the corresponding ball of the standard capped infinite cylinder.

(c) *For $A_1 > 0$, the Sobolev embedding*

$$\left(\int v^{2n/(n-2)}\,\mathrm{d}\mu\big(g(T_1)\big)\right)^{(n-2)/n} \leqslant A_1 \int \big(4|\nabla v|^2 + Rv^2\big)\,\mathrm{d}\mu\big(g(T_1)\big) + A_1 \int v^2\,\mathrm{d}\mu\big(g(T_1)\big)$$

holds for all $v \in W^{1,2}(\mathbf{M}, g(T_1))$.

Then for all $t \in (T_1, T_2]$, the Sobolev embedding below holds for all $v \in W^{1,2}(\mathbf{M}, g(t))$.

$$\left(\int v^{2n/(n-2)}\,\mathrm{d}\mu\big(g(t)\big)\right)^{(n-2)/n} \leqslant A_2 \int \big(4|\nabla v|^2 + Rv^2\big)\,\mathrm{d}\mu\big(g(t)\big) + A_2 \int v^2\,\mathrm{d}\mu\big(g(t)\big).$$

Here

$$A_2 = C\Big(A_1, \sup R^-(x,0), T_2, T_1, \sup_{t\in[T_1,T_2]} \mathrm{Vol}\big(\mathbf{M}\big(g(t)\big)\big)\Big)$$

is independent of the number of surgeries or r.

Moreover, the Ricci flow is strong κ noncollapsed in the whole interval $[T_1, T_2]$ under scale 1 where κ depends only on A_2.

Remark 1.1. By the work Hebey [4], at any given time, a Sobolev embedding always holds with constants depending on lower bound of Ricci curvature and injectivity radius. So one can replace assumption (c) by the assumption that $(\mathbf{M}, g(T_1))$ is κ noncollapsed and that the canonical neighborhood assumption (with a fixed radius $r_0 > 0$ and $\epsilon_0 > 0$) at T_1 holds. It is easy to see that these together imply the Sobolev embedding at time T_1.

We assume as usual that, at a surgery, we throw away all compact components with positive sectional curvature, and also capped horns, double horns and all compact components lying in the region where $R > (\delta r)^{-2}$. In the extra assumption that the Ricci flow is smooth outside of the ϵ horns, we have excluded these deleted items.

Remark 1.2. With the exception of using the monotonicity of Perelman's W entropy, the proof of Theorem 1.1 uses only long established results. Under (r, δ) surgery, assumption (b) is clearly implied by, but much weaker than the *canonical neighborhood assumption* on the whole manifold $\mathbf{M}$, which was used in all the papers so far. In particular there is no need for the gradient estimate on the scalar curvature, which is difficult to prove by it self.

Remark 1.3. In [10], it was shown that under a Ricci flow with finite number of surgeries in finite time, a uniform Sobolev embedding holds. Recently, in the preprint [Y], *The Logarithmic Sobolev inequality along the Ricci flow*, by Ye, Rugang, arXiv: 0707.2424v4, 2007, a similar result depending on the number of surgeries was stated without proof.

Let us finish by outlining the proof. Recall Perelman's W entropy and its monotonicity. They are in fact the monotonicity of the best constants of the Log Sobolev inequality with certain parameters. If a Ricci flow is smooth over a finite time interval, then the best constants of the Log Sobolev inequality with a changing parameter does not decrease. If a Ricci flow undergoes a (r, δ) surgery with δ sufficiently small, then the best constant only decreases by at most a constant times the change in volume. This is achieved by a weighted estimate of Agmon type for the minimizing equation of the W entropy. The method is motivated by those at the end of [9] and [2] where the change of eigenvalues of the linear operator $4\Delta - R$ was studied. Therefore in finite time, the best constant of the Log Sobolev inequality with certain parameters is uniformly bounded from below by a negative constant, regardless of the number of surgeries. This uniform Log Sobolev inequality is then converted by known method to the desired uniform Sobolev inequality which in turn yields strong noncollapsing. Details are available in [11].

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