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Number Theory/Mathematical Analysis

# Arithmetic coherent states and quantization theory

André Unterberger

*Département de mathématiques, FRE 311, Université de Reims, BP 1039, 51687 Reims cedex 2, France*

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**Abstract**

Let $f(z) = q^\kappa \sum_{m\geqslant 0} a_m q^m$, with $q = e^{2i\pi z}$ and $0 < \kappa \leqslant 1$, be a holomorphic modular form of real weight $\tau + 1$, $\tau > -1$, for the group $\Gamma = SL(2, \mathbb{Z})$ and for an arbitrary multiplier; let $\mathfrak{s}_\tau$ be the distribution on the half-line such that $\mathfrak{s}_\tau(t) = \sum_{m\geqslant 0} a_m \delta(t - m - \kappa)$. Let $\mathcal{D}_{\tau+1}$ be the usual realization, in a Hilbert space $\mathcal{H}_{\tau+1}$ of functions on the half–line, of a representation from the projective discrete series of $G = SL(2, \mathbb{R})$ (or the prolongation thereof in the case when $-1 < \tau \leqslant 0$). Then, the set of transforms $\mathfrak{s}_\tau^g = \mathcal{D}_{\tau+1}(g^{-1})\mathfrak{s}_\tau$, $g$ describing any set of representatives of $G$ mod $\Gamma$, can be regarded as a set of coherent states for the representation under study. Analyzing appropriate operators in $\mathcal{H}_{\tau+1}$ by means of their diagonal matrix elements against the distributions $\mathfrak{s}_\tau^g$ brings to light, as a spectral-theoretic density, the convolution $L$-function $L(\bar{f} \otimes f, s)$. Much more can, and will, be said in a forecoming Note in the cases when $\tau = \pm\frac{1}{2}$. ***To cite this article: A. Unterberger, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***



**Résumé**

**États cohérents arithmétiques et quantification.** Soit $f(z) = q^\kappa \sum_{m\geqslant 0} a_m q^m$, avec $q = e^{2i\pi z}$ et $0 < \kappa \leqslant 1$, une forme modulaire holomorphe $f$ de poids réel $\tau + 1$, $\tau > -1$, pour le groupe $\Gamma = SL(2, \mathbb{Z})$ et pour un multiplicateur arbitraire ; soit $\mathfrak{s}_\tau$ la distribution sur la demi-droite telle que $\mathfrak{s}_\tau(t) = \sum_{m\geqslant 0} a_m \delta(t - m - \kappa)$. Soit $\mathcal{D}_{\tau+1}$ une représentation de la série discrète projective de $G = SL(2, \mathbb{R})$ (ou du prolongement de celle-ci dans le cas où $-1 < \tau \leqslant 0$) réalisée, de la manière usuelle, dans un espace de Hilbert $\mathcal{H}_{\tau+1}$ de functions sur la demi-droite. Alors, l'ensemble des transformées $\mathfrak{s}_\tau^g = \mathcal{D}_{\tau+1}(g^{-1})\mathfrak{s}_\tau$, $g$ décrivant un système de représentants de $G$ mod $\Gamma$, peut être regardé comme une famille d'états cohérents pour la représentation considérée. L'analyse d'opérateurs appropriés dans $\mathcal{H}_{\tau+1}$ au moyen de leurs éléments de matrices diagonaux contre la famille de distributions $\mathfrak{s}_\tau^g$ fait apparaître, comme densité spectrale, la function $L(\bar{f} \otimes f, s)$. Le cas où $\tau = \pm\frac{1}{2}$ permet davantage et sera traité dans une Note suivante. ***Pour citer cet article : A. Unterberger, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***



Let $\tau > -1$: given $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, and a function $f$ holomorphic in the upper half-plane $\Pi$, set

$$\big(\pi_{\tau+1}(g)f\big)(z) = (-cz + a)^{-\tau-1} f\left(\frac{dz - b}{-cz + a}\right)$$

*E-mail address:* andre.unterberger@univ-reims.fr.

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for some holomorphic determination, for fixed $g$, of the function $z \mapsto (-cz+a)^{-\tau-1}$: which determination has been chosen will turn out to be irrelevant in the sequel. A $\tau$-adapted distribution $\mathfrak{s}_\tau$ is a measure on the line of the kind

$$\mathfrak{s}_\tau(t) = \sum_{m\geqslant 0} a_m \delta(t-m-\kappa)$$

for some $\kappa \in ]0,1]$, where the sequence $(a_m)$ is controlled by some power of $m+1$, satisfying the following property. Set $f(z) = q^\kappa \sum_{m\geqslant 0} a_m q^m$ with $q = \mathrm{e}^{2\mathrm{i}\pi z}$ and $q^\kappa = \mathrm{e}^{2\mathrm{i}\pi\kappa z}$: it is assumed that the function $f$ is invariant, up to the multiplication by some complex number of absolute value 1, by the transformation $\pi_{\tau+1}\big(\big(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\big)\big)$. As an example, one can take $\kappa = \frac{\tau+1}{12}$ and borrow the coefficients $a_m$ from the Fourier expansion of the function

$$f(z) = \big(\eta(z)\big)^{2\tau+2} = q^{\frac{\tau+1}{12}} \prod_{n\geqslant 1} (1-q^n)^{2\tau+2},$$

a fractional power of Ramanujan's $\Delta$-function. We shall also consider [1, p. 249] the convolution $L$-function

$$L(\bar f \otimes f, s) = \sum_{m\geqslant 0} |a_m|^2 (m+\kappa)^{-s-\tau}, \quad \operatorname{Re} s > 1,$$

and the analytic continuation for $s \neq 0, 1$ of the product of this function by $\zeta(2s)$, as provided by the Rankin–Selberg unfolding method.

Consider on the other hand the Hilbert space $\mathcal{H}_{\tau+1}$ of functions $v = v(s)$ on $(0,\infty)$, square-integrable with respect to the measure $s^{-\tau}\,\mathrm{d}s$. In the case when $\tau > 0$, the Laplace transformation $v \mapsto f$, with

$$f(z) = (4\pi)^{\frac{\tau}{2}} \big(\Gamma(\tau)\big)^{-\frac{1}{2}} z^{-\tau-1} \int_0^\infty v(s) \mathrm{e}^{-2\mathrm{i}\pi s z^{-1}}\,\mathrm{d}s,$$

is an isometry from $\mathcal{H}_{\tau+1}$ onto the space of holomorphic functions $f$ in $\Pi$, square-integrable with respect to the measure $(\operatorname{Im} z)^{\tau+1}\,\mathrm{d}\mu(z)$, where $\mathrm{d}\mu$ is the standard invariant measure on $\Pi$. Moreover [4, p. 60], it intertwines the projective representation $\pi_{\tau+1}$ with a (projective) representation $\mathcal{D}_{\tau+1}$ of $G$ in $\mathcal{H}_{\tau+1}$, given in particular, if $b > 0$ and $v \in C_0^\infty(]0,\infty[)$, by the equation

$$\big(\mathcal{D}_{\tau+1}(g)v\big)(s) = \mathrm{e}^{-\mathrm{i}\pi\frac{\tau+1}{2}} \frac{2\pi}{b} \int_0^\infty v(t) \left(\frac{s}{t}\right)^{\frac{\tau}{2}} \exp\left(2\mathrm{i}\pi \frac{ds+at}{b}\right) J_\tau\left(\frac{4\pi}{b}\sqrt{st}\right)\mathrm{d}t.$$

This equation makes it possible to extend the definition of $\mathcal{D}_{\tau+1}$ to the case when $-1 < \tau \leqslant 0$, obtaining as a result a projective representation still unitary, but no longer square-integrable.

Starting from the space $C^\infty_{\tau+1}(\mathbb{R}^+)$ of $C^\infty$ vectors of the representation $\mathcal{D}_{\tau+1}$, it is possible, by duality, to extend the operators $\mathcal{D}_{\tau+1}(g)$ to a distribution setting. In particular, using the extension (antilinear with respect to the variable on the left-hand side) of the scalar product $(\,|\,)_{\tau+1}$ on $\mathcal{H}_{\tau+1}$, define the distributions

$$\mathfrak{s}_\tau^g = \mathcal{D}_{\tau+1}(g^{-1})\mathfrak{s}_\tau, \quad g \in \mathrm{SL}(2,\mathbb{R}),$$

by means of the equation $(\mathfrak{s}_\tau^g|v)_{\tau+1} = (\mathfrak{s}_\tau|\mathcal{D}_{\tau+1}(g)v)_{\tau+1}$: it is understood that $(t \mapsto \delta(t-a)|v)_{\tau+1} = a^{-\tau}v(a)$, $a > 0$. Note that the statement that follows does not depend on the choice of any phase factor (a function of $g$ of absolute value 1) which may affect the definition of $\mathcal{D}_{\tau+1}(g)$.

**Theorem 1.** *Normalize the Haar measure* $\mathrm{d}g$ *on* $G$ *as* $\mathrm{d}g = \frac{1}{2\pi}\,\mathrm{d}\mu(z)\,\mathrm{d}\theta$ *if* $x = x + \mathrm{i}y$ *and* $g = \big(\begin{smallmatrix} y^{\frac12} & xy^{-\frac12}\\ 0 & y^{-\frac12}\end{smallmatrix}\big)\big(\begin{smallmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{smallmatrix}\big)$, $0 \leqslant \theta < 2\pi$. *Let* $v \in C^\infty_{\tau+1}(\mathbb{R}^+)$. *One has the identity*

$$\int_{\Gamma\backslash G} \big|\big(\mathfrak{s}_\tau^g|v\big)_{\tau+1}\big|^2\,\mathrm{d}g = \frac{(4\pi)^{\tau+1}}{\Gamma(\tau+1)} \big\| y^{\frac{\tau+1}{2}} f\big\|^2_{L^2(\Gamma\backslash\Pi)} \|v\|^2_{\tau+1},$$

*where* $f$ *is the modular form of weight* $\tau+1$ *associated to* $\mathfrak{s}_\tau$, *so that the function* $z \mapsto y^{\frac{\tau+1}{2}}|f(z)|$ *is automorphic and rapidly decreasing at infinity in the fundamental domain.*

Polarizing this identity makes it possible to write any $v \in C^\infty_{\tau+1}(\mathbb{R}^+)$ as an integral superposition of the distributions $\mathfrak{s}^g_\tau$, which may therefore be considered as a family of (arithmetic) coherent states for the representation $\mathcal{D}_{\tau+1}$.

It is necessary for the following to consider a symbolic calculus of operators acting within the space $\mathcal{H}_{\tau+1}$, covariant under the representation $\mathcal{D}_{\tau+1}$ and the linear action of $G$ on the plane $\mathbb{R}^2$, taken as a phase space for the symbolic calculus in question (the hyperbolic half-plane would not do here). One must introduce the Euler operator $2\mathrm{i}\pi\mathcal{E} = x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi} + 1$ and its functions in the spectral-theoretic sense, and the involution $\mathcal{G}$ such that

$$(\mathcal{G}h)(x,\xi) = 2\int_{\mathbb{R}^2} h(y,\eta)\mathrm{e}^{4\mathrm{i}\pi(x\eta - y\xi)}\,\mathrm{d}y\,\mathrm{d}\eta.$$

The following is a direct definition of the "soft" horocyclic calculus: obtained after some lengthy calculations:

**Definition 1.** Assume $\tau > -\frac{1}{2}$. Given a symbol $h \in \mathcal{S}(\mathbb{R}^2)$, even and changing to its negative under $\mathcal{G}$, the operator $\mathrm{Op}^\tau_{\mathrm{soft}}(h)$ is defined on $C^\infty_0(]0,\infty[)$ by the equation

$$\big(\mathrm{Op}^\tau_{\mathrm{soft}}(h)u\big)(a) = a^\tau \int_0^\infty u(b)\,\mathrm{d}b \int_{\mathbb{R}^2} h(x,\xi)\mathrm{e}^{2\mathrm{i}\pi(a-b)\frac{\xi}{x}}|x|^{2\tau}\left[x^4 - \frac{(a-b)^2}{4}\right] \times \left[\left(x^2 + \frac{1}{2}(\sqrt{a}-\sqrt{b})^2\right)\left(x^2 + \frac{1}{2}(\sqrt{a}+\sqrt{b})^2\right)\right]^{-\tau-\frac{3}{2}} \mathrm{d}x\,\mathrm{d}\xi.$$

Note that the same formula would lead to the zero operator if a $\mathcal{G}$-invariant symbol were used instead. Since the Gamma function is rapidly decreasing at infinity on vertical lines in the complex plane, the calculus $\mathrm{Op}^\tau_{\mathrm{soft}}$ is a softened up version of another symbolic calculus, to be called the isometric horocyclic calculus, linked to it by the equation

$$\mathrm{Op}^\tau_{\mathrm{soft}}(h) = \frac{2^{-\tau}}{\Gamma(\tau+\frac{3}{2})}\mathrm{Op}^\tau_{\mathrm{iso}}\left(\Gamma(1+\mathrm{i}\pi\mathcal{E})\Gamma\left(\tau+\frac{1}{2}-i\pi\mathcal{E}\right)h\right).$$

The isometric horocyclic calculus (introduced in a different way in [4]) establishes an isometry between the $L^2$-space of symbols and the space of Hilbert–Schmidt operators concerned. In the case when $\tau = \mp\frac{1}{2}$, a quadratic transformation on the variable makes it possible to intertwine the representation $\mathcal{D}_{\tau+1}$ with the even (resp. odd) part of the metaplectic representation, and the isometric horocyclic symbol is none other than the Weyl symbol of the corresponding operator $A$ on $L^2(\mathbb{R})$, made unambiguous by the requirement that $A$ and $A^*$ should be zero on the subspace of $L^2(\mathbb{R})$ consisting of functions of the inappropriate parity.

In the following theorem, the assumption that $h$ is radial makes it possible to relax the assumption that $\tau > -\frac{1}{2}$. One can also, in this case, extend the operator $\mathrm{Op}^\tau_{\mathrm{soft}}(h)$ so as to make the scalar product

$$(\mathcal{A}_\tau h)(z) = \big(\mathfrak{s}^g_\tau \,|\, \mathrm{Op}^\tau_{\mathrm{soft}}(h)\mathfrak{s}^g_\tau\big)$$

meaningful: with $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the right-hand side only depends on $z = \frac{a\mathrm{i}+b}{c\mathrm{i}+d}$ because $h$ is radial. Then, $\mathcal{A}_\tau h$ is automorphic because of the assumptions regarding $\mathfrak{s}_\tau$.

**Theorem 2.** *Let $h \in \mathcal{S}(\mathbb{R}^2)$ be a radial function changing to its negative under $\mathcal{G}$, and let $\tau > -1$ be given. Assume that $h = \mathrm{i}\pi\mathcal{E}h_1$ for some radial $\mathcal{G}$-invariant function $h_1 \in \mathcal{S}(\mathbb{R}^2)$. The spectral decomposition of the automorphic function $\mathcal{A}_\tau h$ is given by the formula*

$$(\mathcal{A}_\tau h)(z) = -2^{-\tau-1}\pi^{\frac{1}{2}}\frac{\Gamma(\tau+1)}{\Gamma(\tau+\frac{3}{2})}h(0)\,\mathrm{Res}_{s=1}\big(L(\bar{f}\otimes f,s)\big) + \frac{2\pi^{\frac{1}{2}}}{\mathrm{i}}\int_{\frac{1}{2}-\mathrm{i}\infty}^{\frac{1}{2}+\mathrm{i}\infty} \psi(1-2s)\,\frac{\Gamma(s-\frac{1}{2})\Gamma(\frac{3}{2}-s)\Gamma(s+\tau)}{2^{s+\tau}\Gamma(s)\Gamma(\tau+\frac{3}{2})}\,\frac{L(\bar{f}\otimes f,s)}{\zeta^*(2-2s)}E^*(z,s)\,ds,$$

*where* $E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s)$*, the last factor being the usual non-holomorphic Eisenstein series, and the function* $\psi$ *is defined by the equation*

$$\psi(s) = \frac{1}{2\pi}\int_0^\infty r^{-s} H(r^2)\,\mathrm{d}r, \quad \operatorname{Re} s < 1,$$

*in which* $h(x,\xi) = H(x^2+\xi^2)$*. In particular, no cusp-forms enter it.*

Note that, from results of Shimura [3] (cf. [1, p. 250]), the zeros of the $L$-function of interest on the spectral line $\operatorname{Re} s = \frac{1}{2}$ include the critical zeros of the zeta function whenever $\kappa = 1$, $\tau + 1$ is an even integer and $f$ is a Hecke eigenform for $\Gamma$.

**Corollary 1.** *Under the assumptions of the preceding theorem, and assuming moreover that* $h(0) = 0$*, let* $h^{\mathrm{iso}}$ *be the isometric horocyclic symbol of the operator* $\mathrm{Op}^\tau_{\mathrm{soft}}(h)$*. One has*

$$\|\mathcal{A}_\tau h\|^2_{L^2(\Gamma\backslash\Pi)} = \frac{1}{\pi}\left\| \frac{\zeta(1-2\mathrm{i}\pi\mathcal{E})}{\zeta(-2\mathrm{i}\pi\mathcal{E})}\, L\left(\bar{f}\otimes f, \frac{1}{2} - \mathrm{i}\pi\mathcal{E}\right) h^{\mathrm{iso}} \right\|^2_{L^2(\mathbb{R}^2)}.$$

The main difficulties lie in calculations relative to operators and their symbols. It is also necessary for the proof to extend the Rankin–Selberg unfolding method or, what leads to this effect, the construction of Eisenstein series by the Poincaré summation method: one cannot substitute for the function $z \mapsto y^s$, in the usual definition of Eisenstein series, the Whittaker function $z \mapsto y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi k y)\mathrm{e}^{2\mathrm{i}\pi k x}$ since the series obtained would never converge. However, one has the following possibility, starting from the function (not a generalized eigenfunction of the Laplacian)

$$c_{s,k}(z) = y^{\frac{3}{2}} K_{s-\frac{1}{2}}(2\pi k y)\mathrm{e}^{2\mathrm{i}\pi k x}:$$

**Theorem 3.** *With* $k = 1, 2, \ldots$*, set*

$$\mathfrak{f}_s(z) = \frac{1}{2}\sum_{(m,n)=1}\left(\frac{y}{|mz+n|^2}\right)^{\frac{3}{2}} K_{s-\frac{1}{2}}\left(\frac{2\pi k y}{|mz+n|^2}\right)\exp\left(2\mathrm{i}\pi k \operatorname{Re}\frac{az+b}{mz+n}\right)$$

*with* $an - bm = 1$ *. The series converges when* $\frac{1}{2} < \operatorname{Re} s < 1$*, and the function* $s \mapsto \mathfrak{f}_s(z)$ *so defined is holomorphic. It extends as a meromorphic function in the whole complex plane, with two families of poles, all simple: the ones from the first family are located at points* $s = n + \frac{3}{2} \pm \frac{\mathrm{i}\lambda_j}{2}$ *or* $s = -n - \frac{1}{2} \pm \frac{\mathrm{i}\lambda_j}{2}$ *where* $n = 0, 1, \ldots$ *and* $(\frac{1+\lambda_j^2}{4})$ *is the sequence of eigenvalues of the hyperbolic Laplacian* $\Delta$ *in* $L^2(\Gamma\backslash\Pi)$*; the ones from the second family are to be found within the sequence* $\{-\frac{1}{2} - n, n = 0, 1, \ldots\}$ *or* $\{\frac{3}{2} + n, n = 0, 1, \ldots\}$*. The function*

$$E_k(z,s) = \frac{2\pi k}{1-2s}\left[\mathfrak{f}_{s-1}(z) - \mathfrak{f}_{s+1}(z)\right]$$

*coincides with* $\alpha_k(s)E(z,s)$*, where*

$$\alpha_k(s) = \frac{\pi^{1-s}}{2}\frac{\Gamma(s-\frac{1}{2})\Gamma(\frac{1}{2}-s)}{\Gamma(1-s)\zeta(2-2s)} k^{s-\frac{1}{2}} \sum_{1\leqslant d|k} d^{1-2s}.$$

The proof is based on the use of the non-holomorphic Poincaré–Selberg series introduced in [2].

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