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Number Theory/Mathematical Analysis

The Weyl calculus and the zeta function

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Abstract

Let $G = \mathrm{SL}(2,\mathbb{R})$ and $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Given a finite set S of primes including 2, and $N = 2\prod_{p\in S} p$, one can define a set (ϖ_ρ) of discretely supported measures on the line, parametrized by the set of squares in $(\mathbb{Z}/N\mathbb{Z})^\times$, with the following properties: (i) the linear space generated by the distributions ϖ_ρ is invariant under the metaplectic transformation $\mathrm{Met}(\tilde{g}^{-1})$ associated to any element $\tilde{g}^{-1}$ of the metaplectic group lying above Γ; (ii) for every $\tilde{g}$ in the metaplectic group, lying above $g \in G$, set $\varpi_\rho^{\tilde{g}} = \mathrm{Met}(\tilde{g}^{-1})\varpi_\rho$: then, given $u \in \mathcal{S}(\mathbb{R})$ with the parity determined by $\#S$, the sum $\sum_\rho |\langle \varpi_\rho^{\tilde{g}}, u\rangle|^2$ only depends on the class Γg, and the integral of this expression over $\Gamma\backslash G$ is the product of $\|u\|^2_{L^2(\mathbb{R})}$ by a positive constant. Analyzing an operator $A: \mathcal{S}'(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ by means of its diagonal matrix elements $(\varpi_\rho^{\tilde{g}} \mid A\varpi_\rho^{\tilde{g}})$ brings to light a natural spectral–theoretic role of the family of partial products of the Eulerian expansion of the zeta function. ***To cite this article: A. Unterberger, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Calcul de Weyl et fonction zeta. Soient $G = \mathrm{SL}(2,\mathbb{R})$ and $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Étant donnés un ensemble fini S de nombres premiers comprenant 2, et $N = 2\prod_{p\in S} p$, on peut construire un ensemble (ϖ_ρ) de mesures à support discret sur la droite, indexé par l'ensemble des carrés dans $(\mathbb{Z}/N\mathbb{Z})^\times$, avec les propriétés suivantes : (i) l'espace vectoriel engendré par les distributions ϖ_ρ est invariant par toute transformation métaplectique $\mathrm{Met}(\tilde{g}^{-1})$ au–dessus d'un point de Γ ; (ii) pout tout $\tilde{g}$ appartenant au groupe métaplectique, au–dessus d'un point $g \in G$, posons $\varpi_\rho^{\tilde{g}} = \mathrm{Met}(\tilde{g}^{-1})\varpi_\rho$: alors, étant donnée $u \in \mathcal{S}(\mathbb{R})$, de parité déterminée par le nombre $\#S$, la somme $\sum_\rho |\langle \varpi_\rho^{\tilde{g}}, u\rangle|^2$ ne dépend que de la classe Γg, et l'intégrale de cette expression au–dessus de $\Gamma\backslash G$ est le produit de $\|u\|^2_{L^2(\mathbb{R})}$ par une constante positive. L'analyse d'un opérateur $A: \mathcal{S}'(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ au moyen des produits scalaires $(\varpi_\rho^{\tilde{g}} \mid A\varpi_\rho^{\tilde{g}})$ met en évidence un rôle naturel, comme ingrédient d'une densité spectrale, joué par tout produit partiel du développement eulérien de la fonction zeta. ***Pour citer cet article : A. Unterberger, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Let N be 4 times a squarefree odd positive integer, and let S be the set of primes dividing N; let Λ be the set of μ mod N such that $\mu^2 \equiv 1$ mod $2N$. Set $\alpha_2 = 2, \alpha_p = 1$ for $2 \neq p \in S$, $M_p = p^{-\alpha_p}N$ and $M = \sum_{p\in S} M_p$, so

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that $(M,N)=1$. The canonical isomorphism $(\mathbb{Z}/N\mathbb{Z})^\times \sim (\mathbb{Z}/4\mathbb{Z})^\times \times \prod_{2\neq p\in S}(\mathbb{Z}/p\mathbb{Z})^\times$ restricts as an isomorphism $\Lambda \sim \{\pm 1\}^{\#S}$: there is a unique character $\chi : \Lambda \to \{\pm 1\}$ the restriction of which to each factor of this decomposition is the non-trivial character of $\{\pm 1\}$. Next, we denote as R_N a set of representatives of $(\mathbb{Z}/N\mathbb{Z})^\times$ mod Λ, choosing in particular $R_4 = R_{12} = \{1\}$. For every $\rho \in R_N$, consider the distribution on the line

$$\varpi_\rho(x) = \sum_{\mu\in\Lambda} \chi(\mu) \sum_{\ell\in\mathbb{Z}} \delta\left(x - \frac{N\ell + \rho\mu}{\sqrt{N}}\right),$$

the parity of which is given by the number $\#S$ of prime factors of N. For instance, the (sole) distribution obtained is

$$\mathfrak{d}_{\text{even}}(x) = \sum_{m\in\mathbb{Z}} \chi^{(12)}(m)\delta\left(x - \frac{m}{\sqrt{12}}\right) \quad \text{if } N = 12,$$

$$\mathfrak{d}_{\text{odd}}(x) = \sum_{m\in\mathbb{Z}} \chi^{(4)}(m)\delta\left(x - \frac{m}{2}\right) \quad \text{if } N = 4,$$

if one denotes as $\chi^{(12)}$ (resp. $\chi^{(4)}$) the non-trivial even Dirichlet character mod 12 (resp. the non-trivial Dirichlet character mod 4). Letting even (resp. odd) functions u on the line correspond to functions v on the half-line under the correspondence such that $u(x) = |x|v(\frac{x^2}{2})$ (resp. $u(x) = v(\frac{x^2}{2})\,\operatorname{sign} x$), next using the Laplace transformation from a function or distribution v on the half-line to the function $f(z) = \int_0^\infty v(t)\mathrm{e}^{2\mathrm{i}\pi tz}\,\mathrm{d}t$ in the upper half-plane, one sees that the function f corresponding to the distribution $\mathfrak{d}_{\text{even}}$ is the Dedekind eta function $f^{(12)}(z) = q^{\frac{1}{24}}\prod_{n\geqslant 1}(1-q^n)$, with $q = \mathrm{e}^{2\mathrm{i}\pi z}$, and that corresponding to the distribution $\mathfrak{d}_{\text{odd}}$ is $f^{(4)} = \frac{1}{2}(f^{(12)})^3$. This provides the link with the situation described in a preceding Note [2].

Proposition 1. *Given N as above, the distributions ϖ_ρ, $\rho \in R_N$, are linearly independent and the space they generate depends only on N; it is stable under the multiplication by $\mathrm{e}^{\mathrm{i}\pi x^2}$ as well as under the Fourier transformation. In the basis made up by the above set of distributions, these two transformations are represented by the matrices T and K, where T is the diagonal matrix with entries $\mathrm{e}^{\frac{\mathrm{i}\pi\rho^2}{N}}$ and K is the matrix with entries*

$$K(\rho,\sigma) = N^{-\frac{1}{2}} \prod_{p\in S} \left(\iota_p^{\rho\sigma} - \iota_p^{-\rho\sigma}\right),$$

where $\iota_p = \exp(2\mathrm{i}\pi \frac{\overline{M}}{p^{\alpha_p}})$, denoting as $\overline{M}$ the inverse of M mod N.

Theorem 1. *For every $\tilde{g}$ in the metaplectic group, lying above $g \in G$, set $\varpi_\rho^{\tilde{g}} = \mathrm{Met}(\tilde{g}^{-1})\varpi_\rho$, and let $u \in \mathcal{S}(\mathbb{R})$ be a function with the same parity as the number $\#S$. Denoting (abusively) as $|\langle \varpi_\rho^g, u\rangle|$ the absolute value $|\langle \varpi_\rho^{\tilde{g}}, u\rangle|$, the sum $\sum_{\rho\in R_N} |\langle \varpi_\rho^g, u\rangle|^2$ only depends on the class Γg. Moreover, one has*

$$\sum_{\rho\in R_N} \int_{\Gamma\backslash G} \left|\left\langle \varpi_\rho^g, u\right\rangle\right|^2 dg = \frac{2\pi}{3} N^{-\frac{1}{2}} \phi(N) \|u\|^2_{L^2(\mathbb{R})}, \tag{1}$$

where ϕ is Euler's indicator function, and the Haar measure dg is normalized as in [2].

The polarized version of this equation shows that every function $u \in \mathcal{S}(\mathbb{R})$ with the appropriate parity can be written as an integral superposition of distributions $\varpi_\rho^{\tilde{g}}$, $\rho \in R_N$: we may therefore consider this family as a family of coherent states for the part of the metaplectic representation corresponding to the parity indicated by $\#S$, though of course the distributions in question do not lie in $L^2(\mathbb{R})$.

Given a distribution $\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^2)$, recall that the operator $\mathrm{Op}(\mathfrak{S})$ with Weyl symbol $\mathfrak{S}$ is the linear operator: $\mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ defined by the equation

$$\big(\mathrm{Op}(\mathfrak{S})u\big)(x) = \int_{\mathbb{R}^2} \mathfrak{S}\left(\frac{x+y}{2}, \eta\right) \mathrm{e}^{2\mathrm{i}\pi(x-y)\eta} u(y)\,\mathrm{d}y\,\mathrm{d}\eta.$$

Theorem 2. *Denote as* $2i\pi\mathcal{E}$ *the operator* $x\frac{\partial}{\partial x}+\xi\frac{\partial}{\partial\xi}+1$ *on the plane with coordinates* (x,ξ)*. The distribution* W_N *which is the Weyl symbol of the operator* $u\mapsto\sum_{\rho\in R_N}\langle\varpi_\rho,u\rangle\varpi_\rho$ *is given by the equation*

$$W_N=N^{i\pi\mathcal{E}}\big(\zeta_N(2i\pi\mathcal{E})\big)^{-1}\mathfrak{D}_0,$$

where

$$\zeta_N(s)=\prod_{p\in S}\big(1-p^{-s}\big)^{-1}$$

and $\mathfrak{D}_0$ *is the standard Dirac comb, with support* $\mathbb{Z}\times\mathbb{Z}$*, in the plane.*

Assume that $h\in\mathcal{S}(\mathbb{R}^2)$ is a radial symbol: then, for any $\tilde{g}$ in the metaplectic group lying above some point $g\in G$, the sum $\sum_{\rho\in R_N}(\varpi_\rho^{\tilde{g}}\,|\,\mathrm{Op}(h)\varpi_\rho^{\tilde{g}})$ depends only on $z=g.i$, the point of the upper half-plane Π which is the image of i under the fractional-linear transformation associated to g, and we denote it as $(\mathcal{A}h)^{(N)}(z)$. Thanks to Proposition 1, the function $(\mathcal{A}h)^{(N)}$ is automorphic with respect to the full modular group, and we now determine its spectral (Roelcke–Selberg) decomposition. Towards this, we recall that the Eisenstein series is defined by analytic continuation with respect to s from the equation

$$E(z,s)=\frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)=1}}\left(\frac{|m-nz|^2}{y}\right)^{-s},$$

valid if $\mathrm{Re}\,s>1$, and we set $E^*(z,s)=\pi^{-s}\Gamma(s)\zeta(2s)E(z,s)$, getting as is well-known a function meromorphic in the s-plane with (simple) poles only at $s=0$ or 1, invariant under the symmetry $s\mapsto 1-s$. On the other hand, we set $h(x,\xi)=H(x^2+\xi^2)$ and

$$\psi(s)=\frac{1}{2\pi}\int_0^\infty r^{-s}H\big(r^2\big)\,dr,\qquad \mathrm{Re}\,s<1.$$

Finally, we define the rescaled symplectic Fourier transformation $\mathcal{G}$ in the plane by the equation

$$(\mathcal{G}h)(y,\eta)=2\int_{\mathbb{R}^2}h(x,\xi)e^{4i\pi(y\xi-x\eta)}\,dx\,d\xi:$$

then, even symbols invariant under $\mathcal{G}$ are exactly the symbols of operators which kill odd functions and transform each function into an even one, while even symbols transforming to their negatives under $\mathcal{G}$ correspond to operators doing precisely the opposite so far as parity is concerned.

Theorem 3. *Under the conditions that precede, assume moreover that* $h=\mathcal{G}h$ *if* $\#S$ *is even, and* $h=-\mathcal{G}h$ *if* $\#S$ *is odd. Then, one has*

$$(\mathcal{A}h)^{(N)}(z)=2\int_{-\infty}^{\infty}\frac{N^{-\frac{i\lambda}{2}}}{\zeta_N(-i\lambda)}\frac{\pi^{\frac{1-i\lambda}{2}}}{\Gamma(\frac{1-i\lambda}{2})}\psi(i\lambda)E^*\left(z,\frac{1-i\lambda}{2}\right)d\lambda+(-1)^{\#S}\frac{\phi(N)}{N^{\frac{1}{2}}}h(0)$$

for every $z\in\Pi$*. If, moreover,* $h(0)=0$*, one has the identity*

$$\big\|(\mathcal{A}h)^{(N)}\big\|^2_{L^2(\Gamma\backslash\Pi)}=\frac{8}{\pi}\left\|\frac{\zeta(1-2i\pi\mathcal{E})}{\zeta_N(2i\pi\mathcal{E})}h\right\|^2_{L^2(\mathbb{R}^2)}.$$

For the sake of comparison, let us mention the following non-arithmetic result: for z in the upper half-plane Π, set

$$u_z(t)=2^{\frac{1}{4}}(\mathrm{Im}\,z)^{\frac{1}{4}}e^{-i\pi t^2\bar{z}},$$
$$u_z^1(t)=2^{\frac{5}{4}}\pi^{\frac{1}{2}}(\mathrm{Im}\,z)^{\frac{3}{4}}te^{-i\pi t^2\bar{z}},$$

thus defining families of coherent states (in a usual sense, this time) for the even and odd parts of the metaplectic representation. Given $h \in \mathcal{S}'(\mathbb{R}^2)$, set

$$(\mathcal{C}h)_0(z) = \big(u_z \mid \mathrm{Op}(h)u_z\big),$$
$$(\mathcal{C}h)_1(z) = \big(u_z^1 \mid \mathrm{Op}(h)u_z^1\big)$$

for every $z \in \Pi$. Then, if h is a $\mathcal{G}$-invariant even symbol lying in $L^2(\mathbb{R}^2)$, also the image of some function in $L^2(\mathbb{R}^2)$ by the operator $2\mathrm{i}\pi\mathcal{E}$, the function $(\mathcal{C}h)_0$ lies in $L^2(\Pi)$ and one has

$$\big\|(\mathcal{C}h)_0\big\|_{L^2(\Pi)} = 2\big\|\Gamma(\mathrm{i}\pi\mathcal{E})h\big\|_{L^2(\mathbb{R}^2)}.$$

If h is an even symbol lying in $L^2(\mathbb{R}^2)$ changing to its negative under $\mathcal{G}$, the function $(\mathcal{C}h)_1$ lies in $L^2(\Pi)$ and one has

$$\big\|(\mathcal{C}h)_1\big\|_{L^2(\Pi)} = 4\big\|\Gamma(1+\mathrm{i}\pi\mathcal{E})h\big\|_{L^2(\mathbb{R}^2)}.$$

It is natural to ask what can remain of what precedes in the case when one replaces, say, the distribution $\mathfrak{d}_{\text{even}}$ by the standard Dirac comb $\mathfrak{d}_0$, supported in $\mathbb{Z}$. To start with, one must replace the full modular group Γ by its subgroup Γ_2 generated by the matrices $\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$, which is conjugate within Γ to the Hecke subgroup generally denoted as $\Gamma_0(2)$. As a fundamental domain of the action of that group in Π, one may take the set of points z such that $|z| > 1$ and $|\mathrm{Re}\, z| < 1$: there are two cusps, the cusp $\mathfrak{a}$ corresponding to the point $i\infty$, and the cusp $\mathfrak{b}$ corresponding to any of the points ± 1. There is no analogue of Theorem 1 in that case: the integral in place of the left-hand side of (1) is divergent, which can be traced to the fact that the standard theta function (a holomorphic modular form of weight $\frac{1}{2}$) is not parabolic.

Still, we may set $(\mathcal{B}h)(z) = (\mathrm{Met}(\tilde{g}^{-1})\mathfrak{d}_0 \mid \mathrm{Op}(h)\mathrm{Met}(\tilde{g}^{-1})\mathfrak{d}_0)$ if $\tilde{g}$ is a point of the metaplectic group lying above some $g \in G$ and $z = g.i$, and ask for the spectral decomposition of the Γ_2-automorphic function $\mathcal{B}h$. Following the method [1] of dealing with congruence subgroups, we set, for $\mathrm{Re}\, s > 1$,

$$E_{\mathfrak{a}}(z,s) = \frac{1}{2}\sum_{\substack{(c,d)=1\\ cd\equiv 0 \bmod 2}} \left(\frac{\frac{y}{2}}{|cz+d|^2}\right)^s,$$
$$E_{\mathfrak{b}}(z,s) = \frac{1}{2}\sum_{\substack{(c,d)=1\\ c,d \text{ odd}}} \left(\frac{y}{|cz+d|^2}\right)^s.$$

Theorem 4. *Under assumptions relative to h identical to those of Theorem* 3*, also assuming that $h = \mathcal{G}h$ and* (*for simplicity*) *that $h(0) = 0$, one has*

$$(\mathcal{B}h)(z) = \int_{-\infty}^{\infty} \zeta(1-\mathrm{i}\lambda)\psi(\mathrm{i}\lambda)\left[2^{\frac{3-3\mathrm{i}\lambda}{2}} E_{\mathfrak{a}}\left(z, \frac{1-\mathrm{i}\lambda}{2}\right) + \big(2-2^{1-\mathrm{i}\lambda}\big)E_{\mathfrak{b}}\left(z, \frac{1-\mathrm{i}\lambda}{2}\right)\right]\mathrm{d}\lambda.$$

The results of the present Note are much easier to prove than those of [2], so far as analysis, at least, is concerned. Of course, in the case when $N = 12$ or 4, it is easy to verify that they coincide with the special cases of the results of [2] corresponding to the values $\mp\frac{1}{2}$ of τ. The proofs of the results contained in the two Notes will appear in a volume *Quantization and Arithmetic* with Birkhäuser.

References

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