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Probability Theory

Large gains in the St. Petersburg game

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Abstract

We consider the accumulated gains of geometric size in the St. Petersburg game and study the logarithmic tail asymptotics of their distribution. ***To cite this article: G. Stoica, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Grands gains dans le jeu de Saint Pétersbourg. Nous considérons des gains de taille géométrique accumulés dans le jeu de Saint Pétersbourg et étudions le comportement asymptotique de la queue de leur distribution. ***Pour citer cet article : G. Stoica, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction and results

A single trial of the St. Petersburg game consists in tossing a fair coin until it first lands heads and the player wins 2^k dollars if this happens on the kth toss. Hence, if X is the gain at a single trial, we have

$$P\big(X=2^k\big)=2^{-k} \quad \text{and} \quad P(X>c)=2^{-[\log_2 c]}, \tag{1}$$

for $k=1,2,\ldots$ and $c\geqslant 1$, where $\log_2$ stands for the logarithm to the base 2, and $[x]$ denotes the largest integer not exceeding x.

Let $(X_n)_{n\geqslant 1}$ be i.i.d. random variables distributed like X, representing the player's gains in a sequence of independent repetitions of the St. Petersburg game. Although the expectation is seen to be infinite, for the total winnings $S_n=X_1+X_2+\cdots+X_n$ in n games, Feller [5] proved that $S_n/(n\log_2 n)\to 1$ in probability as $n\to\infty$. It was subsequently shown that, with probability one, the set of limit points of $S_n/(n\log_2 n)$ is the interval $[1,\infty)$, cf. Chow and Robbins [2] and Adler [1]. This result was refined by Martin-Löf [9] and Csörgő–Dodunekova [3], who identified the class of subsequential distributional limits of $S_n/n-\log_2 n$, and by Csörgő and Simons [4] who showed that, with probability one, S_n is asymptotic to $n\log_2 n$ if the largest gains are ignored (i.e., the entry fee is fair except for the largest gains). As shown in Vardi [11], this asymptotic equality is very rarely interrupted by a large gain that puts the player ahead for a relatively short period.

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All these limit laws suggest finding the deviation rates of the accumulated gain S_n in the St. Petersburg game. Hu and Nyrhinen [8] and Gantert [6] obtained the following result for the polynomial size gains, that easily implies the deviation rate for Feller's normalization, or the iterated logarithm normalization, as in Adler [1] or Vardi [11]: for $\varepsilon > 0$ and $b > 1$, one has

$$\lim_{n\to\infty} \frac{\log_2 P(S_n > \varepsilon x_n)}{\log_2 n} = 1 - b, \tag{2}$$

where x_n is any of the following sequences: n^b, $(n\log_2 n)^b$, $(n\log_2\log_2 n)^b$, $n\log_2 n/(\log_2\log_2 n)^{b-1}$.

However, formula (2) does not hold when x_n increases faster than a polynomial; it is our purpose to give an answer to this problem, by looking at geometric size deviations for the distributions of the accumulated gain. The asymptotics in this case are rather different than (2), as we shall see in the following result:

Theorem 1. *Let $(X_n)_{n\geqslant 1}$ be the player's gains in independent St. Petersburg games and let $S_n = X_1 + X_2 + \cdots + X_n$. Then, for $\varepsilon > 0$ and $b > 1$, we have*

$$\lim_{n\to\infty} \frac{\log_2 P[S_n > \varepsilon b^n]}{n} = -\log_2 b. \tag{3}$$

Corollary 1. *With the same notations as above, and if $M_n = \max\{X_1, X_2, \ldots, X_n\}$ denotes the maximal gain in n St. Petersburg games, then*

$$\lim_{n\to\infty} \frac{\log_2 P\big[M_n > \varepsilon b^n\big]}{n} = -\log_2 b.$$

1.1. Interpretation and application

Gains of geometric size were introduced by Martin-Löf [9] and Csörgő–Dodunekova [3]; they proved a central limit theorem (CLT) along geometric subsequences b^n such that the fractional part of $\log_2(b^n)$ converges as $n\to\infty$. In the case $b = 2$, their CLT says that the premium per game $2^m + n$ has only a small probability ($\simeq 1.8\times 2^{-m}$) of being insufficient to cover the accumulated gains S_{2^n}. Our result says that gains of geometric size, as in the above CLT, have probability of occurrence of order b^{-n}. In addition, Corollary 1 quantifies Steinhaus' solution to the St. Petersburg paradox (cf. [10]). He proposed the following geometric sequence of entrance fees at the nth repetition of a St. Petersburg game:

$$2\quad 4\quad 2\quad 8\quad 2\quad 4\quad 2\quad 16\quad 2\quad 4\quad 2\quad 8\quad 2\quad 4\quad 2\quad 32\quad 2\quad 4\quad 2\ldots$$

(place twos in alternating empty places, then fill every second empty place by a four, next fill every second remaining place by an eight, etc.). Steinhaus [10] proved that, with probability one, the sequence of actual gains will have the same distribution as the above sequence. Corollary 1 gives the precise tail asymptotics of Steinhaus' sequence.

2. Proofs

In the sequel we denote by C a strictly positive constant whose value may differ from line to line, even within the same line, and does not depend on n.

Lemma 1 *(cf. Gut [7], Lemma 2.2). Let $(a_n)_{n\geqslant 1}$ be a sequence of real numbers such that $0 \leqslant a_n < C$ for $n \geqslant 1$. Then $(1-a_n)^n \to 1$ as $n\to\infty$ if and only if $na_n \to 0$ as $n\to\infty$. In either case,*

$$1 - (1-a_n)^n \geqslant Cna_n \quad \textit{for n large enough.}$$

Lemma 2. *With the notations in Theorem 1 we have*

$$\lim_{n\to\infty} \frac{\log_2 P[X > \varepsilon b^n]}{\log_2(\varepsilon b^n)} = -1.$$

Proof. By (1) we have: $1/c \leqslant P(X > c) < 2/c$ for $c \geqslant 1$. Apply $\log_2$ in the latter inequalities, then divide by $\log_2 c$ and take $c := \varepsilon b^n$ with n large enough so that $c > 1$; finally let $n \to \infty$. □

Remark that $\sup\{d > 0\colon\ E(X^d) < \infty\} = 1$; in particular $E(X^{1/b})$ is finite for any $b > 1$, and we have the following:

Lemma 3 *(cf. Hu and Nyrhinen [8], Lemma 3.2). Let $b > 1$ be fixed. Then, for any $t, s > 0$ and natural $n \geqslant 1$, we have*

$$P\big(S_n > t^b\big) \leqslant n P\left[X > \left(\frac{t}{s}\right)^b\right] + e^s E^{s^b}\big(X^{1/b}\big)\left(\frac{n}{t}\right)^{s^b},$$

where E is the symbol for expectation under P.

Proof of Theorem 1. Let $\delta > 0$ be fixed. By Lemma 2 we have

$$\log_2 P\big[X > \varepsilon b^n\big] \geqslant -(1+\delta)\log_2(\varepsilon b^n) \quad \text{for large } n. \tag{4}$$

Recall that $M_n = \max\{X_1, X_2, \ldots, X_n\}$; we deduce:

$$\begin{aligned} P\big[S_n > \varepsilon b^n\big] &\geqslant P\big[M_n > \varepsilon b^n\big] \quad \text{as } X, X_1, X_2, \ldots \text{ are nonnegative} \\ &= 1 - \big\{1 - P\big[X > \varepsilon b^n\big]\big\}^n \quad \text{as } X, X_1, X_2, \ldots \text{ are i.i.d.} \\ &\geqslant 1 - \big\{1 - \big[\varepsilon b^n\big]^{-1-\delta}\big\}^n \quad \text{for large } n \text{ (using (4))} \\ &\geqslant Cn\big[\varepsilon b^n\big]^{-1-\delta} \quad \text{for large } n \text{ by Lemma 1 with } a_n = \big[\varepsilon b^n\big]^{-1-\delta}. \end{aligned}$$

Apply $\log_2$ in the above inequalities and obtain:

$$\log_2 P\big[S_n > \varepsilon b^n\big] \geqslant C + \log_2 n - (1+\delta)\log_2\big(\varepsilon b^n\big) \quad \text{for large } n;$$

as $\delta > 0$ is arbitrary, the latter inequality gives

$$\log_2 P\big[S_n > \varepsilon b^n\big] \geqslant C + \log_2 n - n\log_2 b \text{ for large } n;$$

divide by n and obtain

$$\liminf_{n\to\infty} \frac{\log_2 P[S_n > \varepsilon b^n]}{n} \geqslant -\log_2 b. \tag{5}$$

On the other hand, let us take $t = (\varepsilon b^n)^{1/b}$ in Lemma 3; using again (1) we obtain for all $n \geqslant 1$:

$$\begin{aligned} P\big[S_n > \varepsilon b^n\big] &\leqslant n P\left[X > \frac{\varepsilon b^n}{s^b}\right] + e^s E^{s^b}\big(X^{1/b}\big)\left(\frac{n}{(\varepsilon b^n)^{1/b}}\right)^{s^b} \\ &\leqslant \frac{2ns^b}{\varepsilon b^n} + \frac{e^s E^{s^b}(X^{1/b})}{\varepsilon^{s^b/b}}\big(nb^{-n/b}\big)^{s^b}. \end{aligned}$$

Apply $\log_2$ in the latter inequality, use that $\log_2(x+y) \leqslant 1 + \max\{\log_2 x, \log_2 y\}$ for all $x, y > 0$, and deduce that, for all $n \geqslant 1$, we have:

$$\log_2 P\big[S_n > \varepsilon b^n\big] \leqslant 1 + \max\big\{C + \log_2\big(nb^{-n}\big), C + s^b\log_2\big(nb^{-n/b}\big)\big\}.$$

Further divide by n and obtain

$$\begin{aligned} \limsup_{n\to\infty} \frac{\log_2 P\big[S_n > \varepsilon b^n\big]}{n} &\leqslant \limsup_{n\to\infty} \frac{1}{n}\cdot\max\left\{\log_2 n - n\log_2 b, s^b\left(\log_2 n - \frac{n}{b}\log_2 b\right)\right\} \\ &= \max\left\{-\log_2 b, -s^b\frac{\log_2 b}{b}\right\} \\ &= -\log_2 b \end{aligned} \tag{6}$$

(the last equality follows by letting s increase until the second term in the latter maximum becomes smaller than $-\log_2 b$). Finally, relations (5) and (6) prove (3). □

Proof of Corollary 1. As estimation from below of $P[M_n > \varepsilon b^n]$ is already obtained just below formula (4), hence we obtain formula (5) with S_n replaced by M_n. On the other hand, as $M_n \leqslant S_n$, as estimation from above of $P[M_n > \varepsilon b^n]$ is given by formula (6). The rest is straightforward. □

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