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Complex Analysis/Functional Analysis

Expansion in series of exponential polynomials of mean-periodic functions with growth conditions

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Abstract

Let θ be a Young function. Consider the space $\mathcal{F}_\theta(\mathbb{C})$ of all entire functions on $\mathbb{C}$ with θ-exponential growth. In this Note, we are interested in the solutions $f \in \mathcal{F}_\theta(\mathbb{C})$ of the convolution equation $T \star f = 0$, called T-mean-periodic functions, where T is in the topological dual of $\mathcal{F}_\theta(\mathbb{C})$. We show that each mean-periodic function admits an expansion as a convergent series of exponential polynomials. ***To cite this article: H. Ouerdiane, M. Ounaies, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Développement en séries de polynômes exponentiels des fonctions moyenne-périodiques à croissance $\boldsymbol{\theta}$-exponentielle. Soit θ une fonction de Young. Considérons l'espace $\mathcal{F}_\theta(\mathbb{C})$ de toutes les fonctions entières sur $\mathbb{C}$ à croissance θ-exponentielle. On s'intéresse dans cette Note aux solutions $f \in \mathcal{F}_\theta(\mathbb{C})$ de l'équation de convolution $T \star f = 0$, appelées fonctions T-moyenne-périodiques, où T est dans le dual topologique de $\mathcal{F}_\theta(\mathbb{C})$. On montre que toute fonction moyenne-périodique admet un développement en série de polynômes exponentiels. De plus cette série est convergente pour la topologie de l'espace $\mathcal{F}_\theta(\mathbb{C})$. ***Pour citer cet article : H. Ouerdiane, M. Ounaies, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***
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Version française abrégée

Notons $\mathcal{H}(\mathbb{C})$ l'espace des fonctions entières sur $\mathbb{C}$. Soit $\theta : [0, +\infty] \to [0, +\infty]$ une fonction de Young, c'est-à-dire que θ est convexe, continue, croissante et vérifie $\theta(0) = 0$ et $\lim \frac{x}{\theta(x)} = 0$ quand $x \to +\infty$. Notons θ^* sa transformée de Legendre définie par $\theta^*(x) = \sup_{t \geqslant 0}(tx - \theta(t))$, $x \geqslant 0$. Nous considérons l'espace, noté $\mathcal{F}_\theta(\mathbb{C})$, des fonctions $f \in \mathcal{H}(\mathbb{C})$ vérifiant la condition de croissance

$$\forall m > 0, \quad \sup_{z\in\mathbb{C}} \left| f(z) \right| e^{-\theta^*(m|z|)} < \infty, \tag{1}$$

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et nous notons $\mathcal{F}'_\theta(\mathbb{C})$ son dual topologique fort. Soit $T \in \mathcal{F}'_\theta(\mathbb{C})$, $T \neq 0$. Nous dirons que $f \in \mathcal{F}_\theta(\mathbb{C})$ est T-moyenne périodique si elle vérifie l'équation de convolution $T \star f = 0$.

Soit $\mathcal{L}$ la transformation de Fourier–Borel définie sur $\mathcal{F}'_\theta(\mathbb{C})$. On note $\{\alpha_k\}_k$ les zéros de $\mathcal{L}(T)$ et m_k leur ordre de multiplicité. Les monômes exponentiels $z^l e^{\alpha_k z}$, $0 \leqslant l < m_k$, sont alors des fonctions T-moyennes périodiques. Notre résultat principal est le suivant :

Théorème 0.1. *Toute fonction T-moyenne-périodique f est série convergente dans $\mathcal{F}_\theta(\mathbb{C})$:*

$$f(z) = \sum_{k \geqslant 1} \sum_{l=0}^{m_k - 1} c_{k,l} \left[\sum_{j=1}^{k} e^{z\alpha_j} P_{k,j,l}(z) \right], \tag{2}$$

où $P_{k,j,l}$ sont les polynômes de degré $< m_j$ donnés par (12) *et* (13)*. Les coefficients $c_{k,l}$ sont donnés au moyen de la transformation de Fourier–Borel $\mathcal{L}$ par* : $c_{k,l} = \langle S_{k,l}, f \rangle$, *avec*

$$\mathcal{L}(S_{k,l})(\xi) = (\xi - \alpha_k)^l \prod_{j=1}^{k-1} (\xi - \alpha_j)^{m_j}.$$

De plus ces coefficients vérifient

$$\forall m > 0, \quad \sum_{k \geqslant 1} e^{\theta(m|\alpha_k|)} \left(\sum_{l=0}^{m_k - 1} |c_{k,l}| |\alpha_k|^{-(m_1 + \cdots + m_{k-1} + l)} \right) < +\infty. \tag{3}$$

Réciproquement, toute série de la forme (2) *dont les coefficients $c_{k,l}$ vérifient* (3) *converge dans $\mathcal{F}_\theta(\mathbb{C})$ vers une fonction T-moyenne-périodique.*

1. Preliminaries and definitions

Let $\theta : [0, +\infty] \to [0, +\infty]$ be a Young function, i.e., θ is convex, continuous, strictly increasing and verifies $\theta(0) = 0$ and $\lim \frac{x}{\theta(x)} = 0$ when $x \to +\infty$. Denote by θ^* the Legendre transform of θ defined by $\theta^*(x) = \sup_{t \geqslant 0}(tx - \theta(t))$, for $x \geqslant 0$, which is also a Young function. Denote by $\mathcal{H}(\mathbb{C})$ the space of all entire functions on the complex space $\mathbb{C}$. For any $m > 0$, consider $E_{\theta,m}(\mathbb{C})$, the Banach space of all functions $f \in \mathcal{H}(\mathbb{C})$ such that

$$\|f\|_{\theta,m} := \sup_{z \in \mathbb{C}} \big| f(z) \big| e^{-\theta(m|z|)} < +\infty \tag{4}$$

and define $\mathcal{G}_\theta(\mathbb{C}) = \bigcup_{p \in \mathbb{N}^*} E_{\theta,p}(\mathbb{C})$ endowed with the inductive limit topology. The space $\mathcal{G}_\theta(\mathbb{C})$ is an algebra under the ordinary multiplication of functions. We define $\mathcal{F}_\theta(\mathbb{C}) = \bigcap_{p \in \mathbb{N}^*} E_{\theta^*,1/p}(\mathbb{C})$ endowed with the projective limit topology and we denote by $\mathcal{F}'_\theta(\mathbb{C})$ the strong topological dual of $\mathcal{F}_\theta(\mathbb{C})$.

For any fixed $u \in \mathbb{C}$, the translation operator τ_u on $\mathcal{F}_\theta(\mathbb{C})$ defined by $(\tau_u f)(z) = f(z + u)$, leaves invariant this space.

For all $S \in \mathcal{F}'_\theta(\mathbb{C})$ and $f \in \mathcal{F}_\theta(\mathbb{C})$, the function $z \to \langle S, \tau_z f \rangle$ is an element of $\mathcal{F}_\theta(\mathbb{C})$. Therefore, for any $S \in \mathcal{F}'_\theta(\mathbb{C})$, the map $S\star : \mathcal{F}_\theta(\mathbb{C}) \to \mathcal{F}_\theta(\mathbb{C})$ defined by $S \star f(z) = \langle S, \tau_z f \rangle$ is a convolution operator, i.e., it is linear, continuous and commute with any translation operator. For any $S \in \mathcal{F}'_\theta(\mathbb{C})$, the Fourier–Borel transform of S, denoted by $\mathcal{L}(S)$ is defined by

$$\mathcal{L}(S)(\xi) = \langle S, e^{\xi \cdot} \rangle,$$

where $e^{\xi \cdot}$ denotes the function $z \to e^{\xi z}$. For any two elements S and U of $\mathcal{F}'_\theta(\mathbb{C})$, the convolution product $S \star U \in \mathcal{F}'_\theta(\mathbb{C})$ is defined by

$$\langle S \star U, f \rangle = \langle S, U \star f \rangle, \quad f \in \mathcal{F}_\theta(\mathbb{C}).$$

Under this convolution, $\mathcal{F}'_\theta(\mathbb{C})$ is a commutative algebra admitting δ_0, the Dirac measure at the origin, as unit. From [5] we deduce the following proposition:

Proposition 1.1. *The space $\mathcal{F}_\theta(\mathbb{C})$ is a nuclear Fréchet space and the Fourier–Borel transform $\mathcal{L}$ is a topological isomorphism between the algebras $\mathcal{F}'_\theta(\mathbb{C})$ and $\mathcal{G}_\theta(\mathbb{C})$.*

2. Main results

Throughout the rest of the paper, let T be a fixed non-zero element of $\mathcal{F}'_\theta(\mathbb{C})$.

Definition 2.1. We say that a function $f \in \mathcal{F}_\theta(\mathbb{C})$ is T-mean-periodic if it satisfies the equation

$$T \star f = 0. \tag{5}$$

Denote by Φ the entire function in $\mathcal{G}_\theta(\mathbb{C})$ defined by $\Phi = \mathcal{L}(T)$.

Remark 2.2. In the case where Φ has no zeros, by a division theorem, we can show that $\frac{1}{\Phi} \in \mathcal{G}_\theta(\mathbb{C})$. Thus, $S = (\mathcal{L})^{-1}(\frac{1}{\Phi}) \in \mathcal{F}'_\theta(\mathbb{C})$. Then, we have $S \star T = T \star S = \delta_0$. If we assume that $T \star f = 0$, then $\delta_0 \star f = f = 0$. Therefore, the only mean-periodic function $f \in \mathcal{F}_\theta(\mathbb{C})$ is the zero function.

We will throughout the rest of the paper assume that Φ has zeros, and denote them by $|\alpha_1| \leqslant |\alpha_2| \leqslant \cdots \leqslant |\alpha_k| \leqslant \cdots$, $\alpha_k \neq \alpha_{k'}$ if $k \neq k'$. We denote by m_k be the order of multiplicity of Φ at α_k.

Remark 2.3. For all $\xi \in \mathbb{C}$ and for all $l \in \mathbb{N}$, the exponential monomial $M_{l,\xi} : z \in \mathbb{C} \to z^l e^{\xi z}$ verifies $\langle T, M_{l,\xi}\rangle = \Phi^{(l)}(\xi)$. In particular, for $0 \leqslant l < m_k$, M_{l,α_k} is a T-mean-periodic function.

Theorem 2.4. (i) *Any T-mean-periodic function $f \in \mathcal{F}_\theta(\mathbb{C})$ admits the following expansion as a convergent series in $\mathcal{F}_\theta(\mathbb{C})$*

$$f(z) = \sum_{k \geqslant 1} \sum_{l=0}^{m_k-1} c_{k,l} \left[\sum_{j=1}^{k} e^{z\alpha_j} P_{k,j,l}(z) \right] \tag{6}$$

where $P_{k,j,l}$ are the polynomials of degree $< m_j$ given by (12) *and* (13)*. The coefficients $c_{k,l}$ verify the following estimate*

$$\forall m > 0, \quad \sum_{k \geqslant 1} e^{\theta(m|\alpha_k|)} \left(\sum_{l=0}^{m_k-1} |c_{k,l}| |\alpha_k|^{-(m_1+\cdots+m_{k-1}+l)} \right) < +\infty \tag{7}$$

and are given by $c_{k,l} = \langle S_{k,l}, f\rangle$ where $S_{k,l} \in \mathcal{F}'_\theta(\mathbb{C})$ is defined by

$$\mathcal{L}(S_{k,l})(\xi) = (\xi - \alpha_k)^l \prod_{j=1}^{k-1} (\xi - \alpha_j)^{m_j}.$$

(ii) *Conversely, any such series whose coefficients $c_{k,l}$ satisfy the estimate* (7) *converges in $\mathcal{F}_\theta(\mathbb{C})$ to a T-mean-periodic function.*

Remark 2.5. Although $\theta(x) = x$ is not a Young function, our results are still valid. In this case, $\mathcal{G}_\theta(\mathbb{C}) = \mathrm{Exp}(\mathbb{C})$, the space of all entire functions of exponential type and the corresponding $\mathcal{F}_\theta(\mathbb{C})$ is the space of all entire functions without any growth condition of the type (4), i.e., $\mathcal{F}_\theta(\mathbb{C}) = \mathcal{H}(\mathbb{C})$.

These results generalize those obtained in [1] and [4], where the authors considered the case $\mathcal{F}_\theta(\mathbb{C}) = \mathcal{H}(\mathbb{C})$. In fact, they showed that, given $T \in \mathcal{H}'(\mathbb{C})$, there exists a sequence of indices $k_1 = 1 < k_2 < \cdots$ such that any T-mean periodic function $f \in \mathcal{H}(\mathbb{C})$ admits a unique expansion, convergent in $\mathcal{H}(\mathbb{C})$, of the form

$$f(z) = \sum_{n \geqslant 1} \sum_{k_n \leqslant k < k_{n+1}} e^{\alpha_k z} \sum_{l=0}^{m_k-1} d_{k,l} \frac{z^l}{l!}. \tag{8}$$

In (8), the sum converges by grouping the terms. Unlike in (6), there is no Abel summation process, but in general, the sequence $\{k_n\}_n$ is not explicit. We refer to [2] for representation formulas of the form (8) in more general cases and to [3] for a general survey on the topic of convolution equations and related problems in interpolation in $\mathbb{C}^n$.

3. Proof of Theorem 2.4

We are going to use a characterization, obtained in [7] of the image of the restriction operator ρ defined on $\mathcal{G}_\theta(\mathbb{C})$ by

$$\rho(g)=\left\{\frac{g^l(\alpha_k)}{l!}\right\}_{k\geqslant 1,\,0\leqslant l<m_k}.$$

This characterization is given in terms of growth conditions involving the divided differences (see [6] for further details about divided differences). To any discrete doubly indexed sequence $a=\{a_{k,l}\}_{k,\,0\leqslant l<m_k}$ of complex numbers, we associate the sequence of divided differences $\Psi(a)=b=\{b_{k,l}\}_{k,\,0\leqslant l<m_k}$. We recall that they are the coefficients of the Newton polynomials

$$Q_q(\xi)=\sum_{k=1}^{q}\prod_{j=1}^{k-1}(\xi-\alpha_j)^{m_j}\left(\sum_{l=0}^{m_k-1}b_{k,l}(\xi-\alpha_k)^l\right), \tag{9}$$

defined, for any $q\geqslant 1$, as the unique polynomial of degree $m_1+\cdots+m_q-1$ such that

$$\frac{Q_q^{(l)}(\alpha_k)}{l!}=a_{k,l},\quad 1\leqslant k\leqslant q \text{ and } 0\leqslant l\leqslant m_k-1.$$

When all the multiplicities $m_k=1$, we may give a simple formula for the coefficients b_k:

$$b_k=\sum_{j=1}^{k}a_j\prod_{1\leqslant n\leqslant k,\,j\neq n}(\alpha_j-\alpha_n)^{-1}.$$

Let us denote by $\mathcal{B}_{\theta,m}(V)$ the Banach space of all doubly indexed sequences of complex numbers $b=\{b_{k,l}\}_{k,\,0\leqslant l<m_k}$ such that,

$$\|b\|_{\theta,m}:=\sup_{k\geqslant 1}\sup_{0\leqslant l<m_k}|b_{k,l}||\alpha_k|^{m_1+\cdots+m_{k-1}+l}\mathrm{e}^{-\theta(m|\alpha_k|)}<+\infty \tag{10}$$

and by $\mathcal{A}_{\theta,m}(V)=\Psi^{-1}(\mathcal{B}_{\theta,m}(V))$. The space $\mathcal{A}_{\theta,m}(V)$ endowed with the norm $\|a\|_{m,\theta}=\|\Psi(a)\|_{m,\theta}$ is a Banach space and Ψ is an isometry from $\mathcal{A}_{\theta,m}(V)$ into $\mathcal{B}_{\theta,m}(V)$.

Now, consider the spaces $\mathcal{A}_\theta(V)=\bigcup_{p\in\mathbb{N}^*}\mathcal{A}_{\theta,p}(V)$ and $\mathcal{B}_\theta(V)=\bigcup_{p\in N^*}\mathcal{B}_{\theta,p}(V)$ endowed with the topology of inductive limit of Banach spaces.

In [7], we showed that the map $\rho:\mathcal{G}_\theta(\mathbb{C})\to\mathcal{A}_\theta(V)$ is continuous and surjective. Therefore, the linear map $\alpha=\Psi\circ\rho\circ\mathcal{L}:\mathcal{F}_\theta'(\mathbb{C})\to\mathcal{B}_\theta(V)$ is also continuous and surjective.

The space $\mathcal{F}_\theta(\mathbb{C})$ is a Fréchet–Schwartz space, therefore it is reflexive. Then, the transpose α^t of α is defined from the strong dual of $\mathcal{B}_\theta(V)$, denoted by $\mathcal{B}_\theta'(V)$, into $\mathcal{F}_\theta(\mathbb{C})$. From a classical theorem (see [1]), α^t is a topological isomorphism onto its image and $\operatorname{Im}\alpha^t=(\operatorname{Ker}\alpha)^\circ$, the orthogonal space of $\operatorname{Ker}\alpha$.

Lemma 3.1. *The space $\mathcal{B}_\theta'(V)$ can be identified, through the canonical bilinear form*

$$\langle c,b\rangle=\sum_{k=1}^{+\infty}\sum_{l=0}^{m_k-1}c_{k,l}b_{k,l}$$

with the space $\mathcal{C}_\theta(V)=\bigcap_{p\in\mathbb{N}^*}\mathcal{C}_{\theta,p}(V)$ endowed with the projective limit topology, where, for all p, $\mathcal{C}_{\theta,p}(V)$ is the Banach space of the sequences $c=\{c_{k,l}\}_{k,\,0\leqslant l<m_k}$ such that

$$\|c\|_{\theta,p}':=\sum_{k\geqslant 1}\mathrm{e}^{\theta(p|\alpha_k|)}\left(\sum_{l=0}^{m_k-1}|c_{k,l}||\alpha_k|^{-(m_1+\cdots+m_{k-1}+l)}\right)<+\infty. \tag{11}$$

Moreover, $\mathcal{C}_\theta(V)$ is a Fréchet–Schwartz space.

Proof. For the first part of the lemma, it suffices to see that, for all $p \in \mathbb{N}^*$, for all $b \in \mathcal{B}_{\theta,p}(V)$ and all $c \in \mathcal{C}_{\theta,p}(V)$,

$$\big|\langle c, b\rangle\big| \leqslant \|c\|'_{\theta,p}\|b\|_{\theta,p}.$$

For the second part, it is clear that, for any $p \in \mathbb{N}^*$, the canonical injection $i_p : \mathcal{C}_{\theta,p+1}(V) \to \mathcal{C}_{\theta,p}(V)$ is compact. Thus, in view of [1, Proposition 1.4.8.], $\mathcal{C}_\theta(V)$ is a Fréchet–Schwartz space. □

Lemma 3.2. *We have the equalities*: $\operatorname{Im}\alpha^t = (\operatorname{Ker}\alpha)^\circ = \{f \in \mathcal{F}_\theta(\mathbb{C}) \mid T \star f = 0\}$.

Proof. As $\operatorname{Ker}\rho$ is the ideal generated by Φ in $\mathcal{G}_\theta(\mathbb{C})\}$, we deduce that $\operatorname{Ker}\alpha$ is the ideal generated by T in $\mathcal{F}'_\theta(\mathbb{C})$. Let f be an element of $(\operatorname{Ker}\alpha)^\circ$. For all $z \in \mathbb{C}$,

$$(T \star f)(z) = \langle T, \tau_z f\rangle = \langle T, \delta_z \star f\rangle = \langle T \star \delta_z, f\rangle = 0,$$

using the fact that $T \star \delta_z \in \operatorname{Ker}\alpha$.

Conversely, let $f \in \mathcal{F}_\theta(\mathbb{C})$ be such that $T \star f = 0$ and let $U \in \mathcal{F}'_\theta(\mathbb{C})$. We have

$$\langle T \star U, f\rangle = \langle U, T \star f\rangle = 0.$$

This shows that $f \in(\operatorname{Ker}\alpha)^\circ$ and concludes the proof of the lemma. □

Let us proceed with the proof of Theorem 2.4, (i). Let $f \in \mathcal{F}_\theta(\mathbb{C})$ be a T-mean periodic function. From Lemmas 3.2 and 3.1, there is a unique sequence $c \in \mathcal{C}_\theta(V)$ such that $f = \alpha^t(c)$. For $z \in \mathbb{C}$, denoting by δ_z the Dirac measure at z, we have

$$f(z) = \langle\delta_z, f\rangle = \big\langle\delta_z, \alpha^t(c)\big\rangle = \big\langle c, \alpha(\delta_z)\big\rangle = \big\langle c, \Psi\big(\rho(g_z)\big)\big\rangle$$

where we have denoted by $g_z = \mathcal{L}(\delta_z)$, that is, the function in $\mathcal{G}_\theta(\mathbb{C})$ defined by $g_z(\xi) = \mathrm{e}^{z\xi}$. Let us compute $\Psi(\rho(g_z)) = b(z) = \{b_{k,l}(z)\}_{k,\,0\leqslant l<m_k}$, which is an element of $\mathcal{B}_\theta(V)$. By well know formulas about Newton polynomials (see, for example [1, Definition 6.2.8]), we have, for $k \in \mathbb{N}^*$, and denoting by $\partial_j^m = \frac{1}{m!}\frac{\partial^m}{\partial\alpha_j^m}$, for $0 \leqslant l < m_k$,

$$b_{k,l}(z) = \partial_1^{m_1-1}\cdots\partial_{k-1}^{m_{k-1}-1}\partial_k^l\left(\sum_{j=1}^k \mathrm{e}^{z\alpha_j}\prod_{1\leqslant n\leqslant k,\, n\neq j}(\alpha_j-\alpha_n)^{-1}\right) = \sum_{j=1}^k \mathrm{e}^{z\alpha_j}P_{k,j,l}(z),$$

where we have denoted by, for $1 \leqslant j \leqslant k-1$,

$$P_{k,j,l}(z) = \sum_{i=0}^{m_j-1}\frac{z^i}{i!}\partial_j^{m_j-1-i}\left(\prod_{1\leqslant n\leqslant k-1,\, n\neq j}(\alpha_j-\alpha_n)^{-m_n}(\alpha_j-\alpha_k)^{-(l+1)}\right) \quad \text{and} \tag{12}$$

$$P_{k,k,l}(z) = \sum_{i=0}^{l}\frac{z^i}{i!}\partial_k^{l-i}\left(\prod_{1\leqslant n\leqslant k-1}(\alpha_k-\alpha_n)^{-m_n}\right). \tag{13}$$

Thus, we obtain

$$f(z) = \sum_{k\geqslant 1}\sum_{l=0}^{m_k-1} c_{k,l}b_{k,l}(z) = \sum_{k\geqslant 1}\sum_{l=0}^{m_k-1} c_{k,l}\left[\sum_{j=1}^k \mathrm{e}^{z\alpha_j}P_{k,j,l}(z)\right]$$

and the equality (6) is established. For any $p \in \mathbb{N}^*$, observe that $\|g_z\|_{\theta,p} \leqslant \mathrm{e}^{\theta^*(\frac{1}{p}|z|)}$. By continuity of $\Psi \circ \rho$, there exists $p' \in \mathbb{N}^*$ and $C_p > 0$ such that

$$\big\|b(z)\big\|_{\theta,p'} \leqslant C_p\|g_z\|_{\theta,p} \leqslant C_p\mathrm{e}^{\theta^*(\frac{1}{p}|z|)}.$$

We deduce, using (10) and (11), that

$$\sup_{z\in\mathbb{C}}\sum_{k\geqslant 1}\sum_{l=0}^{m_k-1}\big|c_{k,l}b_{k,l}(z)\big|\mathrm{e}^{-\theta^*(\frac{1}{p}|z|)} \leqslant C_p\sum_{k\geqslant 1}\mathrm{e}^{\theta(p'|\alpha_k|)}\left(\sum_{l=0}^{m_k-1}|c_{k,l}||\alpha_k|^{-(m_1+\cdots+m_{k-1}+l)}\right) = C_p\|c\|'_{\theta,p'}.$$

It is now clear that the right-hand side of (6) is absolutely convergent in $\mathcal{F}_\theta(\mathbb{C})$. Now, for fixed k and l with $0 \leqslant l < m_k$, consider the element $b^{k,l} = \{\delta_{kj}\delta_{li}\}_{j,\,0\leqslant i<m_k}$ of $\mathcal{B}_\theta(\mathbb{C})$. The corresponding Newton polynomials (9) are given by $Q_q = 0$ when $q < k$ and $Q_q = S_{k,l}$ when $q \geqslant k$. Consequently, we see that $\alpha(S_{k,l}) = \Psi \circ \rho \circ \mathcal{L}(S_{k,l}) = b^{k,l}$ and

$$\langle S_{k,l}, f\rangle = \langle S_{k,l}, \alpha^t(c)\rangle = \langle \alpha(S_{k,l}), c\rangle = \langle b^{k,l}, c\rangle = c_{k,l}.$$

(ii) The converse part is easily deduced from the previous estimates and Remark 2.3.

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