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Mathematical Analysis

A characterization of the Fourier transform and related topics ☆

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Abstract

It is shown that the Fourier transform is essentially, up to a simple adjustment, the only transform on the corresponding space which maps convolutions to products and products to convolutions (surprisingly, no linearity is assumed a priori). ***To cite this article: S. Alesker et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

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Résumé

Une caractérisation de la transformation de Fourier et questions connexes. On montre que la transformation de Fourier est essentiellement, à une simple adaptation près, la seule application, qui sur les espaces où elle opère, transforme les convolutions en produits et les produits en convolutions. (De manière surprenante la linéarité n'est pas supposée à priori.) ***Pour citer cet article : S. Alesker et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

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To state the various results we need to recall first some well known definitions and simple observations. For a reference on the standard definitions and results stated below, see e.g. [3], the elementary introduction to the subject [5], or the more advanced [4].

One says that an infinitely smooth function $f:\mathbb{R}^n \to \mathbb{C}$ is *rapidly decreasing* (also called *Schwartz function*) if for any $l \in \mathbb{Z}_+$ and any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ of non-negative integers one has

$$\sup_{x\in\mathbb{R}^n}\left|\frac{\partial^\alpha f(x)}{\partial x^\alpha}\left(1+|x|^l\right)\right| < \infty,$$

where as usual $\frac{\partial^\alpha f(x)}{\partial x^\alpha} := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$, $|\alpha| := \sum_{i=1}^n \alpha_i$.

The space of all complex valued rapidly decreasing functions on $\mathbb{R}^n$ is denoted by $\mathcal{S}$, and is called the Schwartz space.

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The space $\mathcal{S}$ becomes a Fréchet space when equipped with the system of norms:

$$\|f\|_N := \sup\left\{\left|\frac{\partial^\alpha f(x)}{\partial x^\alpha}\left(1+|x|^N\right)\right| \;\middle|\; x \in \mathbb{R}^n, |\alpha| \leqslant N\right\}.$$

One of the main properties of the Schwartz space $\mathcal{S}$ is that the Fourier transform $\mathbb{F}: \mathcal{S} \to \mathcal{S}$, defined by,

$$(\mathbb{F}f)(\xi) = \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-2\pi \mathrm{i}\langle x,\xi\rangle}\,\mathrm{d}x,$$

is a linear topological isomorphism.

The space $\mathcal{S}$ has two structures of algebra given by the point-wise product and the convolution:

$$\cdot: \mathcal{S} \times \mathcal{S} \to \mathcal{S},$$
$$*: \mathcal{S} \times \mathcal{S} \to \mathcal{S}.$$

Both operations are continuous with respect to both arguments simultaneously.

Let $\mathcal{S}'$ be the topological dual of $\mathcal{S}$. Elements of $\mathcal{S}'$ are called *distributions of tempered growth*. We have the canonical continuous map $\mathcal{S} \to \mathcal{S}'$ given by:

$$\langle \phi, f\rangle = \int_{\mathbb{R}^n} f(x)\phi(x)\,\mathrm{d}x.$$

This map is injective and has a dense image in the weak topology. We will identify $\mathcal{S}$ with its image in $\mathcal{S}'$: $\mathcal{S} \subset \mathcal{S}'$.

The following claim is well known:

Claim 1. *Let $\mathcal{S}$ be equipped with the above Fréchet topology, and $\mathcal{S}'$ be equipped with the weak topology.*

(i) *The point-wise product on $\mathcal{S}$ extends* (*uniquely*) *to a separately continuous map $\mathcal{S} \times \mathcal{S}' \to \mathcal{S}'$ which is given explicitly by,*

$$\langle \phi, \psi \cdot f\rangle = \langle \phi \cdot \psi, f\rangle,$$

for any $f \in \mathcal{S}'$, $\phi, \psi \in \mathcal{S}$. Then $\mathcal{S}'$ becomes a module over $\mathcal{S}$.

(ii) *The convolution on $\mathcal{S}$ extends* (*uniquely*) *to a separately continuous map $\mathcal{S} \times \mathcal{S}' \to \mathcal{S}'$ which is given explicitly by,*

$$\langle \phi, \psi * f\rangle = \left\langle \phi * (-\mathrm{Id})^*\psi, f\right\rangle,$$

for any $f \in \mathcal{S}'$, $\phi, \psi \in \mathcal{S}$. Then $\mathcal{S}'$ becomes a module over $\mathcal{S}$. (*Here* $(-\mathrm{Id})^*$ *denotes the operator defined by* $((-\mathrm{Id})^*\psi)(x) = \psi(-x)$.)

(iii) *The Fourier transform extends* (*uniquely*) *to an isomorphism of linear topological spaces $\mathbb{F}: \mathcal{S}' \tilde{\to} \mathcal{S}'$. Moreover $\mathbb{F}$ is an isomorphism of $\mathcal{S}$-modules: for any $\phi \in \mathcal{S}$, $f \in \mathcal{S}'$,*

$$\mathbb{F}(\phi \cdot f) = \mathbb{F}(\phi) * \mathbb{F}(f),$$
$$\mathbb{F}(\phi * f) = \mathbb{F}(\phi) \cdot \mathbb{F}(f).$$

For a proof see [3], Chapter 2.

To give a sample of our main results, we start with one which is not the strongest, but which we find to be a true 'characterization of the Fourier transform':

Theorem 2. *Assume we are given a bijective transform $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ which admits an extension $\mathcal{F}': \mathcal{S}' \to \mathcal{S}'$ which is also bijective, and such that for every $f \in \mathcal{S}$ and $g \in \mathcal{S}'$ we have $\mathcal{F}'(f \cdot g) = (\mathcal{F}f) * (\mathcal{F}'g)$. Assume also that $\mathcal{F}' \circ \mathcal{F}' = (-\mathrm{Id})^*$ (that is, for every $f \in \mathcal{S}'$, $\mathcal{F}'(\mathcal{F}'(f))(x) = f(-x)$).*

Then, there exists $B \in \mathrm{GL}_n(\mathbb{R})$ with $B = B^*$ and $|\det(B)| = 1$ such that either for every $f \in \mathcal{S}$, $\mathcal{F}f = \mathbb{F}(f \circ B)$, or, for every $f \in \mathcal{S}$, $\mathcal{F}f = \overline{\mathbb{F}(f \circ B)}$.

In particular we see that the theorem implies that a transform satisfying the condition *must* be linear and continuous (in any reasonable sense) which we did not assume a-priori.

Note that if we add the assumption that $\mathcal{F}' \circ \mathcal{F}' = (-\mathrm{Id})^*$, then conditions implies that not only are products mapped to convolutions, but also vise-versa. Thus, with this formally stronger condition, the theorem would be a consequence of the following:

Theorem 3. *Assume we are given a bijective transform $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ which admits an extension $\mathcal{F}': \mathcal{S}' \to \mathcal{S}'$ which is also bijective, and such that*

1. *For every $f \in \mathcal{S}$ and $g \in \mathcal{S}'$ we have $\mathcal{F}'(f * g) = (\mathcal{F}f) \cdot (\mathcal{F}'g)$,*
2. *For every $f \in \mathcal{S}$ and $g \in \mathcal{S}'$ we have $\mathcal{F}'(f \cdot g) = (\mathcal{F}f) * (\mathcal{F}'g)$.*

Then, $\mathcal{F}$ is essentially the Fourier transform $\mathbb{F}$, that is, for some $B \in \mathrm{GL}_n(\mathbb{R})$ with $|\det(B)| = 1$, we have either for all f that $\mathcal{F}(f) = \mathbb{F}(f \circ B)$ or for all f that $\mathcal{F}(f) = \overline{\mathbb{F}(f \circ B)}$.

This result, in turn, follows from the next one:

Theorem 4. *Assume we are given a bijective transform $\mathcal{T}: \mathcal{S} \to \mathcal{S}$ which admits a bijective extension $\mathcal{T}': \mathcal{S}' \to \mathcal{S}'$ such that*

1. *For every $f \in \mathcal{S}$ and $g \in \mathcal{S}'$ we have $\mathcal{T}'(f * g) = (\mathcal{T}f) * (\mathcal{T}'g)$,*
2. *For every $f \in \mathcal{S}$ and $g \in \mathcal{S}'$ we have $\mathcal{T}'(f \cdot g) = (\mathcal{T}f) \cdot (\mathcal{T}'g)$.*

Then, for all $f \in \mathcal{S}$ we have $\mathcal{T}f = f \circ B$, or for all f we have $\mathcal{T}f = \overline{f \circ B}$, for some $B \in \mathrm{GL}_n(\mathbb{R})$ with $|\det(B)| = 1$.

Let us denote by $\mathcal{S}_{\mathbb{R}}$ the Schwartz space of *real valued* rapidly decreasing functions, and by $\mathcal{S}'_{\mathbb{R}}$ its (real) topological dual. We have the following result:

Theorem 5. *Assume we are given a bijective transform $\mathcal{T}: \mathcal{S}_{\mathbb{R}} \to \mathcal{S}_{\mathbb{R}}$ which admits an extension $\mathcal{T}': \mathcal{S}'_{\mathbb{R}} \to \mathcal{S}'_{\mathbb{R}}$ which also bijective, and such that for every $f \in \mathcal{S}_{\mathbb{R}}$ and $g \in \mathcal{S}'_{\mathbb{R}}$ we have $\mathcal{T}'(f \cdot g) = (\mathcal{T}f) \cdot (\mathcal{T}'g)$.*

Then, there exists some diffeomorphism $u: \mathbb{R}^n \to \mathbb{R}^n$ such that for all $f \in \mathcal{S}_{\mathbb{R}}$, $\mathcal{T}f = f \circ u$.

It turns out that one can prove the following modified version of Theorem 5 along exactly the same lines: everywhere the space of rapidly decreasing functions $\mathcal{S}_{\mathbb{R}}$ can be replaced with the space of compactly supported real valued smooth functions $\mathcal{D}_{\mathbb{R}}$, and the space $\mathcal{S}'_{\mathbb{R}}$ can be replace by the dual space $\mathcal{D}'_{\mathbb{R}}$. With this modification, Theorem 5 can be generalized to arbitrary smooth manifolds instead of $\mathbb{R}^n$ as follows.

Let X be a smooth manifold. Let $\mathcal{D}_{\mathbb{R}}(X)$ denote the space of compactly supported infinitely smooth $\mathbb{R}$-valued functions on X. Let $\mathcal{M}_{\mathbb{R}}(X)$ denote the space of compactly supported infinitely smooth $\mathbb{R}$-valued measures on X. The space $\mathcal{M}_{\mathbb{R}}(X)$ is equipped with the standard linear locally convex topology (which is inductive limit of Fréchet spaces). Let $\mathcal{D}'_{\mathbb{R}}(X)$ denote the (real) topological dual of $\mathcal{M}_{\mathbb{R}}(X)$. Then we have a natural imbedding $\mathcal{D}_{\mathbb{R}}(X) \hookrightarrow \mathcal{D}'_{\mathbb{R}}(X)$ with dense image in the weak topology given by,

$$f \mapsto \left[\phi \mapsto \int_X f \cdot \phi\right].$$

Naturally $\mathcal{D}'_{\mathbb{R}}(X)$ is a $\mathcal{D}_{\mathbb{R}}(X)$-module.

Theorem 6. *Let $\mathcal{T}: \mathcal{D}_{\mathbb{R}}(X) \to \mathcal{D}_{\mathbb{R}}(X)$ be a bijective (not necessarily linear or/and continuous) map which admits a bijective extension $\mathcal{T}: \mathcal{D}'_{\mathbb{R}}(X) \to \mathcal{D}'_{\mathbb{R}}(X)$ such that for every $f \in \mathcal{D}_{\mathbb{R}}(X)$ and $g \in \mathcal{D}'_{\mathbb{R}}(X)$ one has*

$$\mathcal{T}'(f \cdot g) = (\mathcal{T}f) \cdot (\mathcal{T}'g).$$

Then there exists a C^∞-diffeomorphism $u: X \to X$ such that $\mathcal{T}f = f \circ u$ for all $f \in \mathcal{D}_{\mathbb{R}}(X)$.

Note that in each of the results above, linearity of the transform $\mathcal{T}$ is not assumed a-priori, but is a consequence of the multiplicativity assumption.

Let us briefly comment on where these theorems originated from. For reasons connected with the topic of convex analysis, we were interested in the characterization of a very basic concept in convexity: duality and the Legendre transform. In the paper [1] it was shown that the Legendre transform can be characterized as follows: up to linear terms, it is the only involution on the class of convex lower semi-continuous functions on $\mathbb{R}^n$ which reverses the (partial) order of functions. Since the Legendre transform has another special property, namely that it exchanges summation of functions with their inf-convolution (for definitions and details see [2]), this in fact implied that an involution on lower semi-continuous convex functions which reverses order *must* have this special property. It turns out that also the opposite is true, namely any involutive transform (on this class) which exchanges summation with inf-convolution, must reverse order, and, in fact, be up to linear terms the Legendre transform (see [2] for proofs and a discussion). Thus, already at this stage we observed that very minimal basic properties essentially uniquely define some classical transform which traditionally is defined in a concrete, and quite involved form.

It looks very intriguing to determine how far this point of view can be extended. It turns out that also the classical Fourier transform may be defined essentially uniquely by very minimal and basic conditions, namely by the condition of exchanging convolution with product (together with, for example, the form of the square of the transform). This is what we announced in this paper. The methods of proof are different (for Legendre transform convexity is used very strongly), but the ideology is similar, although in the case of Fourier transform the proofs seem to be, for the moment, more involved.

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