

Partial Differential Equations

Solvability of monotone systems of fully nonlinear elliptic PDE's

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Abstract

We study quasimonotone weakly coupled systems of uniformly elliptic equations of Isaac type. We prove results on existence of viscosity solutions of such systems and give a necessary and sufficient condition for such a system to satisfy the comparison principle. *To cite this article: A. Quaas, B. Sirakov, C. R. Acad. Sci. Paris, Ser. I 346 (2008).*

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Résumé

Solvabilité de systèmes monotones d'EDP complètement non-linéaires. On étudie des systèmes quasi-monotones d'équations complètement non-linéaires, uniformément elliptiques, de type Isaac. On obtient des résultats d'existence de solutions du problème de Dirichlet et une condition nécessaire et suffisante pour qu'un tel système satisfasse le principe de comparaison. *Pour citer cet article : A. Quaas, B. Sirakov, C. R. Acad. Sci. Paris, Ser. I 346 (2008).*

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In this Note we consider systems of fully nonlinear elliptic equations

$$H_i(D^2u_1, Du_1, u_1, \dots, u_n, x) = f_i(x) \quad (1)$$

$i = 1, \dots, n$ in a bounded domain $\Omega \subset \mathbb{R}^N$; $n, N \geq 1$.

Our work is motivated by the well-known paper [3], where the theory of viscosity solutions was applied to get existence and uniqueness results for this type of systems. More precisely, in that paper conditions were given under which (1) satisfies a comparison principle (and so it cannot have more than one solution), and it was shown that whenever the system has a subsolution and a supersolution, which are ordered, then Perron's method guarantees the existence of a solution. It is our goal, on the one hand, to extend as far as possible the uniqueness result from [3] in the uniformly elliptic case, that is, to give a necessary and sufficient condition under which the system satisfies a comparison principle, and on the other hand, to give explicit conditions under which ordered sub- and supersolutions can be found. For a large class of systems our results are optimal. Further, we get results of existence of eigenvalues of vector fully nonlinear operators.

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We consider nonlinear operators of Isaac's type, that is, H_k can be a sup-inf of coupled linear operators

$$\sup_{\alpha \in \mathcal{A}_k} \inf_{\beta \in \mathcal{B}_k} \left\{ \sum_{i,j=1}^N a_{ij,k}^{\alpha,\beta}(x) \partial_{ij} u_k + \sum_{i=1}^N b_{i,k}^{\alpha,\beta}(x) \partial_i u_k + \sum_{j=1}^n c_{j,k}^{\alpha,\beta}(x) u_j \right\} \quad (2)$$

where $\mathcal{A}_k, \mathcal{B}_k$ are arbitrary index sets. When $|\mathcal{B}_k| = 1$ in (2) the corresponding sup-operator is usually referred to as Hamilton–Jacobi–Bellman (HJB) operator. These operators are essential tools in control theory and in theory of large deviations, while Isaac's operators are basic in game theory. We refer to [1,3], and to the references in these papers, for a larger list of problems where systems of type (1) appear. Note conditions (H₁)–(H₃) below are satisfied by (2) provided the coefficients $a_{ij,k}^{\alpha,\beta}, b_{i,k}^{\alpha,\beta}, c_{j,k}^{\alpha,\beta}$ are uniformly bounded, the matrices $(a_{ij,k}^{\alpha,\beta})_{i,j}$ are continuous and have strictly positive eigenvalues, and $c_{j,k}^{\alpha,\beta} \geq 0$ for $j \neq k$.

Let \mathcal{M}^+ be the Pucci extremal operator, $\mathcal{M}_{\lambda,\Lambda}^+(M) = \sup_{A \in \mathcal{A}} \text{tr}(AM)$, where $\mathcal{A} \subset \mathcal{S}_N$ denotes the set of matrices whose eigenvalues lie in $[\lambda, \Lambda]$, for some constants $0 < \lambda \leq \Lambda$. For a function f we denote $f_+ = \max\{f, 0\}$, $f_- = f_+ - f_-$. We set, for all $M \in \mathcal{S}_N$, $p \in \mathbb{R}^N$, $u \in \mathbb{R}^n$,

$$F_i^*(M, p, u) = \mathcal{M}_{\lambda,\Lambda}^+(M) + \gamma|p| + \delta_i|u_i| + \sum_{j \neq i} \delta_j(u_j)_+, \quad i = 1, \dots, n,$$

for some $\gamma, \delta_1, \dots, \delta_n \geq 0$. Further, we make use of the following condition, which permits to compare operators. Given two operators $F(M, p, u, x)$, $H(M, p, u, x)$, we say that H satisfies condition (D_F) provided

$$(D_F) \quad -F(N - M, q - p, v - u, x) \leq H(M, p, u, x) - H(N, q, v, x) \leq F(M - N, p - q, u - v, x).$$

We use this condition for operators F which are convex in (M, p, u) (this is equivalent to supposing that F itself satisfies (D_F)). To fix notations, we reserve the letter F (instead of H) for convex operators. Of course F_i^* are convex. Note HJB operators are convex but general Isaac's are not.

We suppose that the operators H_i are uniformly elliptic with bounded measurable coefficients, that is, for some $\gamma, \delta > 0$ and all $M, N \in \mathcal{S}_N$, $p, q \in \mathbb{R}^N$, $u, v \in \mathbb{R}^n$, $x \in \Omega \setminus \mathcal{N}$ (\mathcal{N} is some null set),

(H₁) H_i satisfies $(D_{F_i^*})$, $i = 1, \dots, n$.

Note (H₁) implies that the function $H_i(M, p, u, x)$ is nondecreasing in all variables u_j with $j \neq i$. Under this condition (1) is called quasimonotone.

Since we want to prove existence results, we suppose that

(H₂) $H_i(M, 0, 0, x)$ is continuous in $\mathcal{S}_N \times \overline{\Omega}$.

Viewing to get results on existence of eigenvalues, we are led to suppose that these operators are positively homogeneous

(H₃) $H_i(tM, tp, tu, x) = tH_i(M, p, u, x)$, for all $t \geq 0$, $i = 1, \dots, n$.

As is natural when studying systems of PDE, it is important to describe the coupling in the system, that is, the way it relates the functions u_i to each other. It turns out that this can be done in terms of the functions $c_{ij}(x) := F_i(0, 0, e_j, x)$, where $e_j \in \mathbb{R}^n$ is the vector whose j -th coordinate is 1 and all other coordinates are zero. We set $\mathcal{C}(x) := (c_{ij}(x))_{i,j=1}^n$. As explained in [1], any such matrix can have its lines and columns renumbered in such a way that it is in block triangular form, with each block on the main diagonal being *irreducible*. We recall a $n \times n$ matrix \mathcal{C} is called irreducible provided for any nonempty sets $I, J \subset \{1, \dots, n\}$ such that $I \cap J = \emptyset$ and $I \cup J = \{1, \dots, n\}$, there exist $i_0 \in I$ and $j_0 \in J$ for which $c_{i_0 j_0} \neq 0$ in Ω , that is, $\{x \in \Omega \mid c_{i_0 j_0}(x) > 0\}$ has positive measure. Whenever \mathcal{C} is irreducible, we say that (1) is *fully coupled*. Simply speaking, a system is fully coupled provided it cannot be split into two subsystems, one of which does not depend on the other.

More precisely, we can always renumber the equations in (1) and the components of u in such a way that we can write $\mathcal{C} = (\mathcal{C}_{kl})_{k,l=1}^m$, where $1 \leq m \leq n$, \mathcal{C}_{kl} are $t_k \times t_l$ matrices for some $t_k \leq n$ with $\sum_{k=1}^m t_k = n$, \mathcal{C}_{kk} is an *irreducible*

matrix for all $k = 1, \dots, m$, and $C_{kl} \equiv 0$ in Ω , for all $k, l \in \{1, \dots, m\}$ with $k < l$. Note that $m = 1$ means C itself is irreducible, while $m = n$ means C is in triangular form. We set $s_0 = 0$, $s_k = \sum_{j=1}^k t_j$, and $S_k = \{s_{k-1} + 1, \dots, s_k\}$. For instance, any 1×1 matrix is irreducible. Up to renumbering, when $n = 2$ we divide the set of 2×2 matrices into two parts: matrices $\begin{pmatrix} * & a \\ b & * \end{pmatrix}$ and $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$, where $a, b \neq 0$ and $*$ stands for an arbitrary function. The first of these matrices is irreducible, the second is not.

We make the convention that all (in)equalities between vectors are understood to hold component-wise. Also, when we speak of a solution we mean L^N -viscosity solutions, unless otherwise stated. We refer to [2] for a general review of this notion. Note that viscosity solutions are continuous and that any function in $W_{loc}^{2,N}(\Omega)$ satisfies (1) almost everywhere – such a solution is called strong – if and only if it is a L^N -viscosity solution.

In the following we denote with $\mathcal{F}[\psi]$, $\mathcal{H}[\psi]$ the vectors $(F_i(D^2\psi_i, \psi_i, \psi, x))_{i=1}^n$, resp. $(H_i(D^2\psi_i, \psi_i, \psi, x))_{i=1}^n$, for any vector function ψ . Our first result concerns existence of eigenvalues for fully coupled systems. We define $\lambda_1^+ = \lambda_1^+(\mathcal{F}, \Omega) = \sup\{\lambda \in \mathbb{R} \mid \text{there exists } \psi \in C(\Omega, \mathbb{R}^n) \text{ such that } \psi > 0 \text{ and } \mathcal{F}[\psi] + \lambda\psi \leq 0 \text{ in } \Omega\}$ and, respectively, $\lambda_1^- = \sup\{\lambda \in \mathbb{R} \mid \text{there exists } \psi \in C(\Omega, \mathbb{R}^n) \text{ such that } \psi < 0 \text{ and } \mathcal{F}[\psi] + \lambda\psi \geq 0 \text{ in } \Omega\}$.

Theorem 1. *Suppose that $F_i(M, p, u, x)$ satisfies (H_1) – (H_3) and is convex in (M, p, u) , for all $1 \leq i \leq n$. Assume $C = (F_i(0, 0, e_j, x))_{i,j=1}^n$ is an irreducible matrix. Then*

(a) *there exists vectors $\varphi_1^+, \varphi_1^- \in W_{loc}^{2,q}(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$, $\forall q < \infty$, such that*

$$\begin{aligned} \mathcal{F}[\varphi_1^\pm] + \lambda_1^\pm \varphi_1^\pm &= 0, & \pm\varphi_1 &> 0 & \text{in } \Omega, \\ \pm\varphi_1 &= 0 & & \text{on } \partial\Omega; \end{aligned}$$

(b) *we have $\lambda_1^+ \leq \lambda_1^-$ and \mathcal{F} has no eigenvalues in $(-\infty, \lambda_1^- + \varepsilon) \setminus \{\lambda_1^+, \lambda_1^-\}$, where $\varepsilon > 0$ depends on $N, n, \lambda, \Lambda, \nu, \delta_i$ and Ω ;*

(c) *assume there is a vector $\Psi \in C(\Omega, \mathbb{R}^n)$ such that $\Psi > 0$ and $\mathcal{F}[\Psi] \leq 0$ in Ω . Then either $\lambda_1^+ > 0$ or $\lambda_1^+ = 0$ and $\Psi = \text{const.}\varphi_1^+$. Conversely, if there is a vector $\Psi \in C(\Omega, \mathbb{R}^n)$ such that $\Psi < 0$ and $\mathcal{F}[\Psi] \geq 0$ in Ω then either $\lambda_1^- > 0$ or $\lambda_1^- = 0$ and $\Psi = \text{const.}\varphi_1^-$;*

(d) *if we normalize $\varphi_1^\pm = (\phi_{1,1}^\pm, \dots, \phi_{1,n}^\pm)$ in such a way that $\min_{1 \leq i \leq n} \phi_{1,i}^\pm(x_0) = 1$ for some $x_0 \in \Omega$, then $\sup_\Omega \varphi_1^\pm := \sup_\Omega \max_{1 \leq i \leq n} \phi_{1,i}^\pm \leq C$, where C depends only on $x_0, N, n, \Omega, \lambda, \Lambda, \gamma, \delta$, and on the measures of the subsets of Ω on which $c_{ij} > 0$, where c_{ij} are functions used to define the full coupling in C .*

It follows from this theorem that to any \mathcal{F} such that $F_i(M, p, u, x)$ satisfies (H_1) – (H_3) and is convex in (M, p, u) we can associate a set of numbers $\lambda_{11}^\pm, \dots, \lambda_{1m}^\pm$, where m is the number of irreducible blocks which appear in the decomposition of C , and λ_{1k}^\pm are the eigenvalues, given by Theorem 1, of the subsystem containing only the operators (F_i) with indices $i \in S_k$, in which all $u_j, j \notin S_k$, are set to zero. We denote $\lambda_1^\pm := \min_{1 \leq k \leq m} \lambda_{1k}^\pm$.

The positivity of the eigenvalues turns out to be a necessary and sufficient condition for the vector operator \mathcal{F} to satisfy the comparison principle and the Alexandrov–Bakelman–Pucci inequality, and to be a sufficient condition for the Dirichlet problem (1) to be solvable. We recall that a second order operator \mathcal{H} satisfies the *comparison principle* (CP), provided for any $u, v \in C(\overline{\Omega}, \mathbb{R}^n)$, one of which is in $W_{loc}^{2,N}(\Omega)$, such that $\mathcal{H}[u] \geq \mathcal{H}[v]$ in Ω and $u \leq v$ on $\partial\Omega$, we have $u \leq v$ in Ω . A particular case of (CP) is the *maximum principle*, when one of u, v is set to zero (and $H(0, 0, 0, x) \equiv 0$, as we always assume).

Theorem 2. *Suppose a second-order operator \mathcal{F} is such that $F_i(M, p, u, x)$ satisfies (H_1) – (H_3) and is convex in (M, p, u) , for all $1 \leq i \leq n$. Then $\lambda_1^+(\mathcal{F}) > 0$ is necessary and sufficient for \mathcal{F} to satisfy (CP). Hence, if a second-order operator \mathcal{H} satisfies $(D\mathcal{F})$, then $\lambda_1^+(\mathcal{F}) > 0$ is sufficient for \mathcal{H} to satisfy (CP).*

The next theorem is an extension of the famous Alexandrov–Bakelman–Pucci (ABP) inequality to systems of our type. This inequality is a fundamental result in the theory of elliptic PDE in nondivergence form, and is the core of the solvability and regularity theory for such equations, developed in the late 70’s by Krylov and Safonov. We show the positivity of the eigenvalues is equivalent to the validity of the ABP inequality.

Theorem 3. Suppose the second-order operator \mathcal{F} is such that $F_i(M, p, u, x)$ satisfies (H_1) – (H_3) and is convex in (M, p, u) , for all $1 \leq i \leq n$. If $\lambda_1^-(\mathcal{F}) > 0$ then for any $u \in C(\overline{\Omega}, \mathbb{R})$, $f \in L^N(\Omega, \mathbb{R})$, the inequality $\mathcal{F}[u] \leq f$ implies $\sup_{\Omega} \max_{1 \leq i \leq n} (u_i)_- \leq C(\sup_{\partial\Omega} \max_{1 \leq i \leq n} (u_i)_- + \|\max_{1 \leq i \leq n} (f_i)_+\|_{L^N(\Omega)})$, where C depends on Ω , $N, n, \lambda, \Lambda, \gamma, \delta$, and $\lambda_1^-(\mathcal{F})$. In addition, if $\lambda_1^+(\mathcal{F}) > 0$ then $\mathcal{F}[u] \geq f$ implies

$$\sup_{\Omega} \max_{1 \leq i \leq n} u_i \leq C \left(\sup_{\partial\Omega} \max_{1 \leq i \leq n} (u_i)_+ + \left\| \max_{1 \leq i \leq n} (f_i)_- \right\|_{L^N(\Omega)} \right).$$

We have the following result on solvability of (1):

Theorem 4. Suppose the second-order operator \mathcal{F} is such that $F_i(M, p, u, x)$ satisfies (H_1) – (H_3) and is convex in (M, p, u) , for all $1 \leq i \leq n$, and suppose that the second-order operator \mathcal{H} satisfies $(D_{\mathcal{F}})$ (recall \mathcal{F} always satisfies $(D_{\mathcal{F}})$). If $\lambda_1^+(\mathcal{F}) > 0$ then for any $f \in L^p(\Omega, \mathbb{R}^n)$, $p \geq N$, there exists a viscosity solution $u \in C(\overline{\Omega}, \mathbb{R}^n)$ of $\mathcal{H}[u] = f$ in Ω and $u = 0$ on $\partial\Omega$.

If in addition all $H_i(M, p, u, x)$ are convex in M then u is the unique viscosity solution of this problem, and $u \in W_{\text{loc}}^{2,p}(\Omega, \mathbb{R}^n)$.

If we only know that $\lambda_1^-(\mathcal{F}) > 0$ then for any $f \in L^p(\Omega, \mathbb{R}^n)$, $p \geq N$, such that $f \geq 0$ in Ω , there exists a nonpositive solution of the Dirichlet problem for \mathcal{F} . Note we could always take \mathcal{F} in the last theorem to be the extremal operator from (H_1) , if $\lambda_1^+(\mathcal{F}^*) > 0$; however using $(D_{\mathcal{F}})$ gives a more precise result, in the sense that we may have $\lambda_1^+(\mathcal{F}^*) \leq 0$ but $\lambda_1^+(\mathcal{F}) > 0$ for some “intermediate” \mathcal{F} between \mathcal{H} and \mathcal{F}^* .

It is important for applications to have bounds on the eigenvalues in terms of the operator and the domain. These bounds permit to verify the hypotheses of the previous theorems. Another condition for the positivity of the eigenvalues is contained in Theorem 1(c).

Theorem 5. Suppose the second-order operator \mathcal{F} is such that $F_i(M, p, u, x)$ satisfies (H_1) – (H_3) and is convex in (M, p, u) , for all $1 \leq i \leq n$. Set $\delta = \sum_{i=1}^n \delta_i$, where δ_i are the numbers from (H_1) . Let d and $|\Omega|$ denote respectively the diameter and the Lebesgue measure of Ω , and let R be the radius of the largest ball inscribed in Ω . There exists a positive constant C_1 , which depends only on $N, n, \lambda, \Lambda, \gamma, d$, and a positive constant C_2 , which depends only on $N, n, \lambda, \Lambda, \gamma$ (C_1, C_2 are bounded when these quantities are bounded), such that

$$\frac{1}{C_1 d |\Omega|^{1/N}} - \delta \leq \lambda_1^+(\mathcal{F}) \leq \lambda_1^-(\mathcal{F}) \leq \frac{C_2(1 + \delta)}{R^2}.$$

The proofs of the theorems make essential use of the recent papers [4,5], where corresponding results for scalar fully nonlinear equations were obtained, and [1], where an Alexandrov–Bakelman–Pucci and Harnack type estimates for systems of type (1) were proved.

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